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An Efficient GPS Position Determination Algorithm

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AN EFFICIENT GPS POSITION DETERMINATION ALGORITHM

THESIS

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THESIS

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Carlos R. Colón

Table of Contents

Page

List of Tables

Page

Abstract

The use of direct, or closed-form solutions of the trilateration equations used to obtain the position fix in GPS receivers is investigated. The paper is concerned with the development of an efficient new position determination algorithm that uses the closed-form solution of the trilateration equations and works in the presence of pseudorange measurement noise and for an arbitrary number of satellites. In addition, an initial position guess is not required and good estimation performance is achieved even under high GDOP conditions. A two step GPS position determination algorithm which 1) entails the solution of a linear regression problem and, 2) an update of the solution based on one nonlinear measurement equation is developed. The *closed-form* solution of the linear regression in step 1 provides an estimate of the GPS solution, viz., user position and user clock bias, as well as the estimation error covariance. In the update step only two to three iterations are required as opposed to five iterations which are normally required in the standard iterative least squares algorithm currently used in GPS. The two step algorithm also provides a data driven prediction of the pseudorange measurement noise strength and the estimation error covariance. The mathematical derivation of the novel and efficient solution algorithm for the GPS pseudorange equations using stochastic modeling is validated in a realistic simulation experiment based on 5000 Monte Carlo runs. The algorithm's performance is discussed and compared to the conventional iterative least squares algorithm currently used in GPS.

1. Introduction

1.1 Background

The NAVSTAR Global Positioning System (GPS) is a space based satellite radio navigation system which provides three dimensional (3-D) user positioning by solving a set of nonlinear trilateration equations using pseudorange measurements. The current method of solving the nonlinear equations is to linearize the pseudorange equations and calculate the user position iteratively, starting with a user provided initial position guess [10]. For near earth navigation, the center of the earth is a good initial guess and the currently used Iterative Least Squares (ILS) algorithm is guaranteed to converge towards the GPS solution. An area of potential improvement that has been investigated in recent years is the use of non-iterative closed-form solutions to the nonlinear pseudorange GPS equations. Closed-form solutions have been developed by Bancroft [19], Krause [11], Abel and Chafee [17], Chafee and Abel [6], Hoshen [12], and by Nardi and Pachter [16], [1].

This thesis is an improvement over [16], [1], by removing a simplification, resulting in a more rigorous mathematical formulation. Indeed, this thesis and references [1] and [16] differ from previous work ([19], [11]) in that an overdetermined system is treated, making use of all in view ($n \geq 6$) satellites as opposed to using just four satellites. Moreover, this research departs from a deterministic formulation of the problem ([19], [6], [12], [11]) and specifically addresses the issue of developing a reliable closed-form solution that works in the presence of measurement noise. Previous work with the exception of reference [20] treated the pseudorange equations as a deterministic set of equations. In this work, it is recognized that pseudorange measurements are noise corrupted. Hence, the stochastic nature of the measurements is reflected in the GPS pseudorange equations from the onset to develop a probabilistically sound GPS solution. By stochastically modeling the measurements situation at hand, solving for position becomes a stochastic estimation problem. The use of correct stochastic modeling and estimation yields a GPS solution that in addition to the position estimate provides an estimate of the measurement noise intensity. Thus, the estimation algorithm developed here provides a data driven position (and user clock bias) estimation error covariance prediction. This introduces a new confidence factor into GPS positioning which is critical for the downstream integration of GPS and an Inertial Navigation System (INS) or SAR sensor. In addition good estimation performance is achieved even under high GDOP conditions. Moreover, direct, or autonomous, solutions which do not require an initial position guess are attractive for space navigation and for unusual planar array configurations using pseudolites, where the iterative process is sensitive to the initial position guess (e.g. the WAAS). Furthermore, fast solutions which do require fewer iterations and Floating-Point Operations (FLOPS) are attractive for high speed vehicles such as spacecraft, because the movement of the vehicle and the rotation of the earth, during the computation interval, have to be accounted for.

The thesis is organized as follows: In Section 2.1 the nonlinear GPS pseudorange equations and the attendant error terms are discussed. A two step GPS position determination procedure is developed. Section 3.1 contains the development of a novel closed-form GPS position estimation algorithm which accounts for measurement noise. The method of linear regression adapted from statistics is used to obtain preliminary closed-form estimates of the position and user clock bias. The number of in-view satellites required is $n \geq 5$. In addition, a data driven estimate of the pseudorange measurement noise intensity is derived. The data driven estimation of the measurement noise intensity requires an additional satellite, thus, the two step algorithm developed in this paper requires at least 6 satellites in-view ($n \geq 6$). In Section 3.2 the second step of the new algorithm is discussed. In step 2 the *closed-form* solution is used in conjunction with one nonlinear measurement equation; thus, an update step, akin to a Kalman-like update technique is developed. This supplementary algorithm uses the solution of the *closed-form* algorithm from Section 3.1 as initialization. The novel two step algorithm is validated in extensive simulations. The experimental setup is discussed in Section 4.1 and the estimation results are given in Section 4.2. Moreover, comparisons are drawn with results achieved using the conventional ILS algorithm currently used in GPS. Good position and clock bias estimates are obtained using the two step algorithm with two to three iterations only, as opposed to five iterations in the ILS algorithm. Also, the FLOPs count is significantly reduced. Conclusions and recommendations are presented in Section 5.1 and 5.2 respectively.

1.2 Research Motivation

The goal in the design of a navigation system is to obtain the best possible positioning accuracy by eliminating or at least minimizing the impact of error sources. The GPS system errors can be attributed to seven basic sources of error: satellite clock errors, atmospheric delays, group delay, ephemeris errors, receiver noise and resolution, multipath errors, and receiver vehicle dynamics [15]. These sources of error are briefly addressed in the discussion of the stochastic modeling of the pseudorange equations in Chapter 3. Although not the only or most intuitive approach to improving positioning accuracy, improvement of the GPS position determination algorithm used within the GPS undoubtedly impacts the achievable accuracy of the solution. Improvements to the algorithm will complement any other system improvements to reduce errors in any of the seven basic error sources. There are numerous potential benefits that encourage improvements to the GPS position determination algorithm; a few that provided the motivation to this thesis are discussed in this section.

1.2.1 Improved User Positioning.

GPS plays an extremely important role in positioning applications, both military and civilian. Regardless of whether precise positioning service (PPS) or standard positioning service (SPS) is being used, an improved GPS position determination algorithm that provides a closed-form solution to the GPS pseudorange equations considering all in-view satellites can improve the accuracy of the GPS position fix. The improvement is expected to be achieved as a result of computing an exact solution to the nonlinear equation as opposed to introducing approximations by linearizing the equations.

1.2.2 Test Range Enhancement.

The Submeter Accuracy Reference System (SARS) navigation test reference system being developed at the 746th Test Squadron at Holoman Air Force Base, is used primarily in the flight testing of integrated navigation systems [18]. With the improvement of GPS, the accuracy of the integrated aircraft navigation systems is also improving. Since the test reference system must provide much higher accuracy than the system being tested, the importance of improved test range accuracy can never be overemphasized. Additionally, he use of an improved GPS position estimation algorithm can enhance the accuracy of the SARS.

1.2.3 Diversified Applications for GPS.

Current iterative GPS algorithms are guaranteed to converge to a correct position solution for near-earth users, initializing the iterations at the earth's center and assuming a zero user clock bias [17]. In space applications, inverted GPS applications such as the SARS [18], [21], and unconventional applications that involve the use of both pseudolites and satellites, the lack of a sufficiently good initial guess may lead to convergence towards the wrong solution. Pseudolite are ground-based transmitters that provide GPS-like positioning data and can be used for augmentation. Inverted GPS applications, and the use of pseudolites are applications that are currently being considered by the 746th Test Squadron for the SARS as a result of work performed at AFIT by McKay [18]. The fact that a closed-form solution will not require an initial position guess is an advantage for these applications.

1.2.4 Position Estimate Error Covariance.

By stochastically modeling the GPS pseudorange equations, solving for position becomes a stochastic estimation problem. The use of correct stochastic modeling and of Kalman Filtering like techniques to solve the estimation problem will lead to a GPS solution that provides accurate estimates of the position estimation error covariance in addition to the position estimate itself. This will introduce a new confidence factor into GPS positioning and is critical for the integration of GPS with additional sensors for integrated navigation systems. This will specifically enhance the accuracy of navigation systems that are integrated using a federated approach but will not help in deep integration schemes as discussed in [22], where the raw pseudorange signals are being used for system integration. Federated system integration is done at processed data levels using computed positioning data from the navigation sensors. This approach is common when system integration is performed as an upgrade on existing sensors that do not provide access to raw data signals required for deep integration [4].

1.2.5 Computational Efficiency.

The use of direct, or closed-form solution algorithms tend to be more computationally efficient than iterative or recursive algorithms. This efficiency results in reduction of the computational cycle which is most advantageous for high speed vehicles. The use of a positioning algorithm with short computation cycles is of extreme importance to space vehicles where the earth's rotation and the vehicle movement, in the computation interval, needs to be taken into consideration.

1.3 Problem Statement

The objective of this thesis is to provide a closed-form solution of the GPS pseudorange equations and presents an effective method for using them in the presence of measurement noise. This thesis work develops an improved closed-form mathematical solution to the GPS pseudorange equations, implements an algorithm based on the mathematical solution, and performs an experimental analysis of the algorithm using realistic Monte-Carlo simulations. The mathematical derivation of the closed-form solution used in this research is closely based on the work of Capt Salvatore Nardi [1] and notes provided by Dr. Meir Pachter [2].

The closed-form positioning solution will consider pseudorange measurements from all available satellites to obtain a position fix. A minimum of 5 satellites will be needed to obtain an initial 3-D position fix using the algorithm developed in this thesis, but this is not a serious problem since there are always more than four satellites in view when the NAVSTAR GPS constellation is fully operational. This thesis work will emphasize the proper use of stochastic modeling and estimation in order to provide appropriate weighting of satellite pseudorange data in producing a positioning solution.

1.4 Scope

The focus of this research is on the improvement of the GPS algorithm to obtain better positioning accuracy and to obtain shorter computational cycles by reducing the number of required iterations. There are numerous other factors in GPS receiver design that affect GPS positioning accuracy that will not be addressed in this thesis. The GPS will be considered at the system level, hence the inner workings of the GPS receiver will not be considered. The GPS pseudoranges that will be used in the research are corrected pseudoranges. The corrected pseudoranges will represent the pseudoranges as they will be provided by the receiver after all known correction factors have been applied and known error modeled out of the raw pseudorange measurements. The correction of these errors are GPS receiver design issues beyond the scope of this thesis. Throughout this thesis, the term pseudoranges is treated as meaning corrected pseudoranges.

1.5 Assumptions

Assumptions must be made about the noise corrupting the pseudorange measurements in order to allow the use of a simple stochastic model and to simplify the stochastic estimation problem. The following assumptions are used in this thesis.

1. After all known corrections are applied to the pseudorange measurements, the residual noise corrupting the pseudorange measurement is a zero-mean Gaussian distributed noise.

2. The noises on all pseudorange measurements have equal variance intensity.

3. The noises on the pseudorange measurements are uncorrelated with one another.

4. The effects of Selective Availability (SA) on pseudorange noise is not considered.

5.A 10 degree elevation angle is always achievable for determination of in-view GPS satellites.

6. The GPS satellite constellation has 24 fully operational satellites.

The extent to which the assumptions are valid is not exactly known; consequently, the impact of using these assumptions can not be determined beforehand. Some of the assumptions made are necessary to obtain the solution to the pseudorange equations. Others are required to establish a realistic baseline for satellite availability used for the experimental simulations. Attempts will be made to qualify the significance of these assumptions on the positioning solution through experimental analysis.

1.6 Methodology

This thesis contains three distinct phases. The first phase is the development of the two step algorithm. The second phase is the experimental implementation of this algorithm. The third phase is the experimental analysis which can only be initiated after the two step algorithm has been completed, implemented, and debugged. The approach will be iterative in nature since rework of the mathematical solution may be required after some experimental analysis is performed, which in turn will require changes to the algorithm and new simulations.

1.7 Pseudorange Modeling in GPS

GPS uses the radio timing principle to measure ranges between the satellites and the GPS receiver making it a time-difference-of-arrival system. If ranges were being measured directly, we would be dealing with a multilateration system and obtaining a position fix would be easy. Under ideal error and noise free conditions, if both the satellite and the GPS receiver's clock were perfectly synchronized on GPS time with no error, then the measured range would be the true range [14]. However, the GPS receiver measures pseudoranges which are corrupted by the receiver clock bias, measurement noise, and other error sources. The latter include atmospheric delays, satellite clock errors, ephemeris errors, and receiver induced errors. The receiver clock bias caused by the difference between the receiver clock time and GPS time is by far the largest contributor to the difference between pseudoranges and ranges; however, the receiver clock bias is common to a set of simultaneous pseudorange measurements making it possible to treat it as an unknown variable to be estimated along with the user position coordinates, hence the GPS solution consist of the user's three space coordinates and his clock time bias.

Ephemeris corrections provided to the satellites from the control segment can be used to partially eliminate the satellite time error and the ephemeris errors. Known tropospheric and ionospheric error model corrections can be applied to partially compensate for tropospheric and ionospheric delay errors. Improved receiver design techniques are used to minimize the effects of the receiver related errors including receiver noise, code loop quantization errors, multipath effects, and interchannel errors. If the residual errors are grouped together under one random variable, v, the GPS pseudorange equation can be modeled as the true Euclidean range with an unknown clock bias and measurement noise superimposed; thus, the stochastic nonlinear pseudorange measurement equation is given by:

$$R_i = \sqrt{(u_x - x_i)^2 + (u_y - y_i)^2 + (u_z - z_i)^2} + b + v_i \tag{1.1}$$

This equation represents the i^{th} corrected pseudorange equation, $i = 1, 2, 3, \ldots, n$, where (u_x,u_y,u_z) are the user position coordinates, (x_i,y_i,z_i) are the known coordinates of the i^{th} satellite, b is the range equivalent user clock bias, v_i is a zero-mean, Gaussian, pseudorange measurement noise, and *n* is the number of satellites in view. It is reasonable to assume that all receiver measurements are subject to the same noise intensity; therefore, they will have the same variance, σ^2. However, the measurement noise terms are not correlated between satellites.

One such measurement equation is available for each of the n in view satellites. These n equations are the GPS pseudorange equations. All positions used in the derivation will be expressed in Earth Centered Earth Fixed (ECEF) coordinates. The nonlinear pseudorange equations will be solved algebraically for the estimated user position, (u_x,u_y,u_z) and user clock bias b. This will be achieved through algebraic manipulation to reduce the GPS pseudorange equations into a linear regression in the form of the standard linear measurement model as defined in [3]. The linear regression is given by

$$\vec{Z} = H\vec{u} + \vec{V} \tag{1.2}$$

where $\vec{Z}$ is the measurement vector, $\vec{u}$ is the vector of unknowns compromising the user position coordinates (u_x,u_y,u_z) and the user clock bias b. **H** is the regressor matrix and $\vec{V}$ is a Gaussian noise vector whose covariance matrix must be determined.

2. Background

This chapter presents the background theory upon which this research will be based. The first part includes a thorough literature review that provides a summary of current knowledge in the field of closed-form GPS algorithms and the application of stochastic modeling to the GPS pseudorange equations. The second part discusses the basic concepts of GPS technology. Particular emphasis is placed on the formulation of the GPS pseudorange equations and the theory behind the current iterative algorithms.

2.1 The Global Positioning System

This section presents an overview of the Global Positioning System to provide some insight into the complexity of the system. This section serves to focus attention to the specific portions of GPS of interest in this thesis, namely the position determination algorithm within the GPS receiver, and shows how it fits into context of the overall GPS system. GPS specific terminology and the coordinate systems used in this thesis are discussed as well.

2.1.1 GPS System Overview.

GPS is a satellite based radio-navigation system that provides worldwide, virtually continuous, three-dimensional (3-D) positioning and accurate timing. The beauty of the system is in its apparent simplicity since, from the user's perspective, extremely accurate positioning can be achieved with the use of a simple, fairly inexpensive GPS receiver. For these reasons, GPS is rapidly becoming the positioning sensor of choice for both military and civilian users. The continuous worldwide coverage provided by GPS makes it ideally suited for air, land, and sea navigation applications. It must be recognized that there is much more to GPS than the portion that the typical user is dealing with in obtaining a GPS position fix. GPS is composed of three segments: the space segment, the control segment, and the user segment. Each segment is essential to the proper functioning of GPS as an accurate and reliable navigation tool. The typical GPS user is concerned with the user segment only.

The GPS space segment refers to the GPS satellite constellation. The GPS constellation consists of 24 satellites, 21 active satellites plus three active spares, in six orbital planes. Each of the satellites has an orbital period of approximately 12 hours (half a sidereal day). A healthy GPS constellation provides satellite coverage such that, for near-earth locations, there are always at least five satellites in-view and at least seven satellites in-view 80 percent of the time [13]. The satellites transmit time-tagged navigation messages which the GPS receivers (user segment) use to calculate their positions. The navigation message information required by the receivers to perform their function, includes GPS time, satellite ephemeris data, correction data, and system almanac data [10].

The control segment is composed of five stations spread over the world, which monitor and control the satellite orbits and GPS time. Only one of the five stations is the master control station and only three of the remaining four stations are uplink stations capable of transmitting data back to the satellites. The five stations receive the same signals seen by all users and collect pseudoranges to all the satellites. All pseudorange measurements collected by the stations are transmitted to the master control station which then computes the true satellite positions and true GPS time. This is possible since the stations are situated at very well surveyed positions. The master control station then calculates corrections for the satellites and transmits them to the uplink stations where the corrections are transmitted to the satellites [1].

The user segment is the GPS receiver. The receiver receives the navigation message, extracts the data and applies the corrections to obtain the pseudorange and pseudorange rate measurements. The user position, velocity, and time are then obtained through an algorithm that calculates a solution from the corrected pseudo-range measurements. The proposed research will specifically address an algorithm that can be used by a GPS receiver to obtain a position fix.

2.1.2 The Earth Centered Earth Fixed (ECEF) Coordinate Frame.

The ECEF coordinate frame is an orthogonal frame with its origin at the earth's center of mass. The ECEF frame is fixed to the earth and therefore rotates with the earth. This frame consists of three axes: x, y, and z. The z axis is aligned with the earth's spin axis directed north and the x and y axes lie in the equatorial plane. The x axis is directed through the Greenwich Meridian (O longitude) and the y axis through the 90 east longitude [10]. ECEF coordinates are commonly used in GPS since, in near earth navigation, the navigator wants to know his positioning with respect to the earth. Calculations in the GPS receiver are normally performed in the ECEF frame for convenience but are converted, in the GPS receiver, to a coordinate system the user selects for display. Geographic coordinates (Latitude, Longitude and Altitude) are commonly used for display purposes.

2.2 *Summary of Current Knowledge*

The proposed thesis research involves the development and evaluation of a closed-form solution to the GPS pseudorange equations using stochastic modeling and estimation. A thorough literature review was required to identify areas and approaches that have not yet been explored in the field of closed-form GPS solutions and to establish the framework for this thesis research. This review of current literature on the NAVSTAR GPS emphasizes the development of closed-form solutions to the pseudorange equations and stochastic estimation.

Relevant papers related to GPS closed-form solutions published over the past twelve years, past theses, articles, and books covering more general aspects of GPS, were reviewed. The information collected is critical to establishing that a problem exists and that the proposed thesis research is a potential solution for improved mathematical modeling for GPS. The literature review covers the following areas: GPS Overview, GPS equations, conventional GPS positioning solutions, and recent alternate approaches to GPS positioning solutions. Upon completion of the literature review, areas of interest that have not yet been addressed will be identified and the viability of pursuing the thesis research will be confirmed.

2.2.1 *The GPS Pseudorange Equations.*

According to Parkinson [5], the GPS pseudorange equation that reflects all the known sources of error is given by

$$R_i \equiv \sqrt{(u_x - x_i)^2 + (u_y - y_i)^2 + (u_z - z_i)^2} + c(b_u - B_i) + c(T + I) + E + W \quad (2.1)$$

where R_i is the raw pseudorange from the user to the i^{th} satellite, (u_x, u_y, u_z) are the user position ECEF coordinates, (x_i, y_i, z_i) are the ECEF coordinates of the i^{th} satellite as calculated from the Keplerian parameters in the satellite's ephemeris data, c is the speed of light in vacuum, b_u, is the receiver clock bias, B_i is the error in satellite time, T is the tropospheric delay, I is ionospheric delay, E are ephemeris errors, and W represents other errors that can attributed to the receiver including receiver noise, code loop quantization error, multipath effects, and interchannel errors.

Ephemeris corrections provided to the satellites from the control segment can be used to eliminate ephemeris errors partially. Known tropospheric and ionospheric error model corrections can be applied to compensate for tropospheric and ionospheric delay errors partially. Improved receiver design techniques are used to minimize the effects of the receiver related errors, including multipath errors. Given the current state of GPS receiver design technology, the residual errors that remain uncompensated can be assumed to be negligibly small. Furthermore, if the residual errors are grouped together under one random variable w the equation reduces to the expression presented as Eq. (1.1) known as the GPS pseudorange equation. If the residual errors are neglected entirely, the ideal GPS pseudorange equation can be expressed as:

$$R_i \approx \sqrt{(u_x - x_i)^2 + (u_y - y_i)^2 + (u_z - z_i)^2} + b \tag{2.2}$$

The ideal GPS pseudorange equation is a nonlinear equation in four unknowns, the three receiver position coordinates, (u_x, u_y, u_z) and the receiver clock bias, b. This equation is the basis for deriving the conventional iterative GPS position solutions. At least four GPS pseudorange equations are required to calculate the four unknowns, (u_x, u_y, u_z, b).

2.2.2 Conventional Iterative Solution to the GPS Pseudorange Equations.

The conventional approach to solving the GPS pseudorange equations is to linearize Eq. (2.2) about a nominal solution for the vector of unknowns. The vector of unknowns, and its associated nominal values, are define as:

$$\vec{u} \equiv [u_x, u_y, u_z, b]^T$$

$$\vec{u}_n \equiv [u_{xn}, u_{yn}, u_{zn}, b_n]^T$$

respectively.

As described in [10], performing a Taylor series expansion of the GPS pseudorange equation and ignoring the second and higher order terms, the following equation is obtained:

$$R_i \approx R_{ni} + \frac{u_{xn} - x_i}{R_{ni} - b_n}\Delta u_x + \frac{u_{yn} - y_i}{R_{ni} - b_n}\Delta u_y + \frac{u_{zn} - z_i}{R_{ni} - b_n}\Delta u_z + \Delta b \tag{2.3}$$

Considering the exactly determined case of four pseudorange equations, the linearized pseudorange equations can be written in matrix form as:

$$\begin{bmatrix} a_{1x} & a_{1y} & a_{1z} & 1 \\ a_{2x} & a_{2y} & a_{2z} & 1 \\ a_{3x} & a_{3y} & a_{3z} & 1 \\ a_{4x} & a_{4y} & a_{4z} & 1 \end{bmatrix} \begin{bmatrix} \Delta u_x \\ \Delta u_y \\ \Delta u_z \\ \Delta b \end{bmatrix} = \begin{bmatrix} R_1 - R_{n1} \\ R_2 - R_{n2} \\ R_3 - R_{n3} \\ R_4 - R_{n4} \end{bmatrix} = \begin{bmatrix} \Delta R_1 \\ \Delta R_2 \\ \Delta R_3 \\ \Delta R_4 \end{bmatrix} \tag{2.4}$$

where,

$$a_{ix} = \frac{u_{xn} - x_i}{R_{ni} - b_n}$$

$$a_{iy} = \frac{u_{yn} - y_i}{R_{ni} - b_n}$$

$$a_{iz} = \frac{u_{zn} - z_i}{R_{ni} - b_n}$$

The a_{ij} entries in the H matrix are recognized as the cosine of the angle between the line-of-sight vector from the receiver to the i^{th} satellite and the j^{th} axis of the ECEF coordinate frame [10].

The linearized pseudorange equations can be written in a more compact form as $H\Delta\vec{u} = \overrightarrow{\Delta R}$. Solving for $\Delta\vec{u}$ gives the result $\Delta\vec{u} = H^{-1}\overrightarrow{\Delta R}$. This equation can be solved iteratively using the following procedure:

1. Estimate an initial $\vec{u}_n$, a nominal receiver position and clock bias.

2. Calculate nominal pseudoranges and difference them with measured pseudoranges to obtain $\overrightarrow{\Delta R}$.

3. Compute direction cosines to form the **H** matrix.

4. Compute $\Delta\vec{u} = H^{-1}\overrightarrow{\Delta R}$.

5. Add $\Delta\vec{u}$ to $\vec{u}_n$, forming a new corrected $\vec{u}_n$, and go back to step 2.

6. Continue process until convergence to a solution is achieved by verifying that $\|\Delta\vec{u}\| \approx 0$ or that an established threshold is attained.

Upon completion, $\vec{u}_n$, the nominal position and bias, represents the best estimate of receiver position and receiver clock bias. This method converges to a solution within three to five iterations even when the initial position guess is nowhere close to the true position [11], for instance the center of the earth. Drawbacks of using this iterative approach include the approximate nature of the linearized equations, computational loading associated with the inversion of a four by four matrix, the requirement for an initial guess, and the possibility of converging to the wrong solution if the initial position guess were not sufficiently close to the true position [11] [17]. The last concern is not an issue for near-earth navigation since a unique solution is guaranteed if the earth's center is used as the initial position guess and a zero initial clock bias is assumed, but is a serious concern if the receiver position is outside the GPS satellite constellation where a unique solution is not guaranteed. The same applies to certain inverted GPS arrangements [25]. To alleviate such concerns, direct closed-form solutions to the GPS pseudorange equations are sought.

2.2.3 Closed-Form Solutions to the GPS Pseudorange Equations.

Although closed-form solutions to the GPS pseudorange equations are attractive, the concept is not new. Joseph Hoshen [12] proposed that a closed-form solution to two-dimensional equations in the form of the GPS pseudorange equations may have been available since the third century BC in the form of the Problem of Apollonius. Since GPS is a fairly recent system, the first article in the open literature concerned with closed-form solutions, specifically tailored to the GPS pseudorange equations, is Stephen Bancroft's in 1985 [19]. Bancroft developed an algebraic solution to the GPS pseudorange equations that was noniterative in nature. His method provides an exact solution in the exactly determined system using four satellites; like the iterative solution, however, it provides a least squares solution in an overdetermined system. The motivation to this solution was accuracy improvement and the possibility of space applications since an initial position-clock bias guess was not required. Bancroft's solution involves solving a quadratic equation, where each of the two roots leads to a potential solution, one of which does not satisfy the pseudorange equations and can be readily eliminated. This solution had a great deal of merit and motivated a number of papers in the years that followed. Driven by accuracy and computational issues including lower dimensionality and speed, Lloyd Krause [11] formulated a direct solution to the GPS pseudorange equation of the determined system based on difference linearization. By differencing the satellite position vectors, a new basis is formed by using any two adjacent difference vectors, forming a measurement plane, and a vector orthogonal to the plane. The four nonlinear pseudorange equations expressed in the new basis are reduced to three linear equations that are independent of the user clock bias and are used to solve for the user position directly. A quadratic auxiliary equation is then formed to solve for the user clock bias. Krause's paper demonstrated a brilliant approach by which differencing is used to linearize quadratic equations and remove dependence on variables. A similar approach will be used in the development of the closed-form algorithm for this thesis research.

Abel and Chaffee [17] demonstrated that in both closed-form solutions presented by Bancroft [19] and by Krause [11], a position fix may not exist and if it does exist, it may not be unique. Abel and Chaffee's paper concluded that, in order to guarantee a unique position fix, an overdetermined system using at least five satellites must be considered. In a subsequent paper [6], Abel and Chaffee suggest that in a pseudorange system such as GPS, the geometry is hyperbolic, unlike the spherical geometry of a ranging system. In a ranging system, ranges are measured directly, unlike the case for GPS in which the pseudoranges include the unknown receiver clock bias. The solution to range equations is obtained geometrically through the intersection of spheres, but this method does not generalize to pseudorange equations because of the unknown bias in each pseudorange; hence, it is not possible to determine the spheres. In view of the fact that the pseudoranges are not only corrupted by an unknown clock bias but also by measurement noise, caution must be taken in dealing with the pseudorange equations when it comes to the use of solutions based on spherical geometry.

2.2.4 Stochastic Modeling.

Pseudorange noise that corrupts the pseudorange measurements is caused by the residual errors discussed earlier. In order to model the GPS pseudorange equations statistically, tremendous effort would have to be dedicated towards the development of reliable noise models. This noise is actually the manifestation of receiver noise and residuals of various measurement errors that remain unmodeled and uncompensated. The major contributors to pseudorange noise that warrant consideration will be discussed.

Although Gaussian-like, receiver noise is better modeled by a longer tailed mixture of Gaussian distributions which can be expressed as:

$$F(x) = (1 - \varepsilon)\Phi(x) + \varepsilon\Phi\left(\frac{x}{3}\right)$$

where Φ is a Gaussian distribution and the parameter ε is generally between 0.01 and 0.1 [23]. In a multichannel receiver, receiver noise can be considered to be uncorrelated across satellites. Consideration must also be given to the receiver clock bias which tends to correlate the pseudorange measurements. The uncompensated residuals of tropospheric and ionospheric errors may have nonzero means that also add to the modeling difficulties. Noise on the satellite position from the ephemeris data will also have some effect on pseudorange noise, but it is likely to be non-Gaussian and have a nonzero mean.

Given the number of contributing factors to pseudorange noise and our lack of knowledge of their characteristics, it is reasonable to propose that the overall pseudorange noise will have a zero-mean Gaussian distribution by invoking the Central Limit Theorem. The Central Limit Theorem states that the sum of many independent random variables, regardless of their distribution, will approach a Gaussian distribution [3]. The Gaussian pseudorange noise will not be white due to the correlated nature of the encompassed errors and noise. This concern is alleviated since there is no requirement for the pseudorange measurements to be uncorrelated in time because the positioning problem will be treated as a static estimation problem, where each snapshot in time is treated as a new static estimation problem. On the other hand, it is desirable to the development of the stochastic estimation problem that pseudorange noise be uncorrelated across satellites. A solid argument for this is not available, but this will not hinder the development of the stochastic closed-form solution to the GPS pseudorange equations in this thesis since the pseudorange measurements will be differenced, thereby eliminating some of the effects of correlated noise. The uncorrelated noise after differencing can only be justified if the effects of Selective Availability (SA) that are highly time correlated in nature are not considered [9]. The choice to overlook the effects of SA in this thesis is not overly restrictive since authorized military users of GPS are not subject to the effects of SA and it is believed that the use of SA will be abolished in the near future.

The assumption that the noise across pseudoranges can be modeled as an independent zero-mean, σ^2 variance random variable is used by Dailey and Bell [20] without any solid justification. The assumption is used to derive statistics for the pseudorange equations position solution errors similar to what is being proposed for this thesis research; however, their approach does not consider a closed-form solution.

2.2.5 The Stochastic Estimation Problem.

The components of an estimation problem are the variables to be estimated, the measurements, and a mathematical model describing relationship between the measurements and the variables to be estimated [3]. Given the lack of dynamics in the GPS pseudorange equations at any given time instant, a static estimation problem can be formulated from the stochastically modeled pseudorange equations. The variables to be estimated and the measurement noise on the pseudoranges can be represented by random variables. The stochastic estimation process will not only provide an optimal estimate of the unknown variables, the user coordinates and user clock bias; but, will also provide the estimation error covariance. This is the most significant motivation to pursuing a stochastic approach to solving the GPS pseudorange equations. However, the error covariance accuracy will be limited by the quality of the measurement noise statistics. In a stochastic estimation problem, this emphasizes the requirement for good noise models. In this research the measurement noise is modeled as a zero-mean, σ^2 variance Gaussian noise, which is believed to be adequate. Due to the lack of knowledge of the variance σ^2

and the fact that it is dependent on receiver design, location, orientation and time of day, attempting to model the noise variance is not a viable option. The approach that is proposed in this research is to determine the variance, (σ^2), of the Gaussian measurement noise, by using the return difference or measurement residual.

2.2.6 Conclusion of Literature Review.

The literature review supports the proposed approach for this thesis research. There are problems associated with the currently used iterative approach to GPS positioning and there is potential for improvement with an exact closed-form solution. With the exception of one recent paper [20], the pseudorange equations have generally been treated as a deterministic set of equations. The lack of effort in the area of stochastic modeling applied to GPS pseudorange equations is evident from the lack of literature on the subject. Previous work on closed-form solutions for the GPS pseudorange equations did not make use of the pseudorange measurements from all in-view satellites; the derivations considered the exactly determined case using only four of the available pseudorange measurements to obtain positioning solutions. Based on the literature review, the development and evaluation of a closed-form solution to the GPS pseudorange equations using stochastic modeling an all in-view satellites, is warranted.

3. Mathematical Derivations

This chapter presents a thorough mathematical derivation of the *closed-form* solution to the GPS pseudorange equations (step 1), and the Kalman-like update equation (step 2).

3.1 Development of a Closed-Form Solution

The mathematical derivation of the *closed-form* solution (step 1) is developed in four parts. The first part presents the algebraic manipulations to transform the stochastic GPS pseudorange equations as shown in Eq. (1.1) into the desired matrix linear regression form as shown in Eq. (1.2). The second part involves the derivation of statistics for the equation error in the linear regression. The third part presents the development of the static stochastic estimator based on a minimum variance estimate that will provide an estimate of the user position coordinates and user clock bias. The final part presents the derivation of the estimation error covariance matrix.

3.1.1 Basic Concepts.

Prior to initiating the mathematical derivation, it is necessary to present some basic concepts and notations that will be used in the sequel:

- $x \sim N(\mu, \sigma^2)$ is the notation used to represent a random variable (x) that has a Gaussian (Normal) probability distribution function with mean (μ) and variance (σ^2).
- E is used to represent the expectation operator. The expectation of a random variable y is given by $E\{y\} = \int_{-\infty}^{\infty} \rho f_y(\rho) d\rho$, where $f_y(\rho)$ is the density of y [3]. This also defines the mean (μ), the first moment of the random variable.

- Expectation is a linear operation; therefore, for any two random variables x and y, then $E\{x+y\} = E\{x\}+E\{y\}$.
- If two random variables x and y are uncorrelated, then $E\{xy\} = E\{x\}E\{y\}$.
- The variable of the i^{th} element of a random vector $\vec{x}$ can be expressed as $P_{ii} \equiv E\left\{(x_i-\mu_i)^2\right\}$, which make up the diagonal elements of the covariance matrix. The off-diagonal elements P_{ij} are zero if the i^{th} and j^{th} elements of the random vector $\vec{x}$ are uncorrelated.
- For a random variable $x \sim N(0, \sigma^2)$, the moments of x are expressed as $E\{x^k\}$ for $k = 1, 2, \ldots, \infty$. An odd k denotes an odd moment and an even k denotes an even moment. All odd moments are zero and even moments are given by $(k-1)\sigma^k$ [26].

3.1.2 Linear Regression.

The corrected pseudorange can be modeled as the true Euclidean range with an unknown clock bias and Gaussian measurement noise superimposed; thus, the stochastic pseudorange measurement equation is given by:

$$R_i = \sqrt{(u_x-x_i)^2+(u_y-y_i)^2+(u_z-z_i)^2}+b+v_i \tag{3.1}$$

This equation represents the i^{th} corrected pseudorange equation, $i = 1, 2, 3, \ldots, n$, where (u_x, u_y, u_z) are the user position coordinates, (x_i, y_i, z_i) are the known coordinates of the i^{th} satellite, b is the range equivalent user clock bias, v_i is a zero-mean, Gaussian, pseudorange measurement noise, and n is the number of satellites in view. It is reasonable to assume that all measurements are subject to the same noise intensity; therefore, they will have the same variance, σ^2. However, the measurement noise terms are not correlated between satellites.

Eq. (3.1) can be written as:

$$(u_x - x_i)^2 + (u_y - y_i)^2 + (u_z - z_i)^2 = (R_i - b - v_i)^2 \tag{3.2}$$

Expanding Eq. (3.2) results in the following equation:

$$\begin{aligned} u_x^2 + u_y^2 + u_z^2 - b^2 - 2x_iu_x - 2y_iu_y - 2z_iu_z + 2R_ib = {} & R_i^2 \\ & - x_i^2 - y_i^2 - z_i^2 - 2R_iv_i + 2bv_i + v_i^2 \end{aligned} \tag{3.3}$$

It is noted that the first four terms in Eq. (3.3) are simply the unknown variables squared and that they are common to all n equations. This presents an opportunity of eliminating the nonlinear terms by differencing; hence, the n^{th} equation is subtracted from the remaining n - 1 equations. The resulting n - 1 equations are linear in the unknown variables and can be expressed as:

$$\begin{aligned} (x_n - x_i)u_x + (y_n - y_i)u_y + (z_n - z_i)u_z + (R_i - R_n)b = {} & \frac{1}{2}\left(v_i^2 - v_n^2\right) \\ + \frac{1}{2}\left(R_i^2 - R_n^2 + x_n^2 - x_i^2 + y_n^2 - y_i^2 + z_n^2 - z_i^2\right) & + R_nv_n - R_iv_i - bv_n + bv_i \end{aligned} \tag{3.4}$$

As a by-product of the preceding operation, the nonlinear n^{th} pseudorange equation remains.

$$R_n = \sqrt{(u_x - x_n)^2 + (u_y - y_n)^2 + (u_z - z_n)^2} + b + v_n \tag{3.5}$$

The n^{th} equation will remain unused for this section of the derivation but will be used subsequently as an auxiliary equation for use in the *Kalman update* solution.

The linear regression in Eq. (3.4) can be compactly written in a matrix notation form as:

$$\vec{Z} = H\vec{u} + \vec{V} \tag{3.6}$$

where:

$$\vec{Z} = \frac{1}{2}\begin{bmatrix} R_1^2 - R_n^2 + x_n^2 - x_1^2 + y_n^2 - y_1^2 + z_n^2 - z_1^2 \\ R_2^2 - R_n^2 + x_n^2 - x_2^2 + y_n^2 - y_2^2 + z_n^2 - z_2^2 \\ \bullet \\ \bullet \\ \bullet \\ R_{n-1}^2 - R_n^2 + x_n^2 - x_{n-1}^2 + y_n^2 - y_{n-1}^2 + z_n^2 - z_{n-1}^2 \end{bmatrix}_{(n-1) \times 1} \tag{3.7}$$

$$H = \begin{bmatrix} (x_n - x_1) & (y_n - y_1) & (z_n - z_1) & (R_1 - R_n) \\ (x_n - x_1) & (y_n - y_1) & (z_n - z_1) & (R_1 - R_n) \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ (x_n - x_1) & (y_n - y_1) & (z_n - z_1) & (R_1 - R_n) \end{bmatrix}_{(n-1) \times 4} \tag{3.8}$$

The parameter $\vec{u} = [u_x, u_y, u_z, b]^T$, and the equation error

$$\vec{V} = \begin{bmatrix} R_n v_n - R_1 v_1 + b(v_1 - v_n) + \frac{1}{2}\left(v_1^2 - v_n^2\right) \\ R_n v_n - R_2 v_2 + b(v_2 - v_n) + \frac{1}{2}\left(v_2^2 - v_n^2\right) \\ \bullet \\ \bullet \\ \bullet \\ R_n v_n - R_{n-1} v_{n-1} + b(v_{n-1} - v_n) + \frac{1}{2}\left(v_{n-1}^2 - v_n^2\right) \end{bmatrix} \tag{3.9}$$

3.1.3 Noise Statistics.

Using the linear regression obtained in Section 3.1.1 as shown in Eq. (3.6), it is possible to obtain an estimate of $\vec{u}$ if the statistics of the noise vector are given. The noise vector statistics are yet to be determined; however, the composition of the noise vector elements is known. The statistics of $\vec{V}$ must be derived from the known statistics of the pseudorange measurement noise v_i.

Statistics for v_i where (i, j = 1,2,...,n):

$$E\{v_i\} = 0$$

$$E\left\{v_i^2\right\} = \sigma^2$$

$$E\{v_i v_j\} = 0 (i \neq j)$$

Statistics for $\vec{V}$ where (i, j = 1,2,..., n-1):

$$\begin{aligned} E\{V_i\} &= E\left\{R_n v_n - R_i v_i + b(v_i - v_n) + \frac{1}{2}\left(v_i^2 - v_n^2\right)\right\} \\ &= R_n E\{v_n\} - R_i E\{v_i\} + b(E\{v_i\} - E\{v_n\}) + \frac{1}{2}\left(E\left\{v_i^2\right\} - E\left\{v_n^2\right\}\right) \\ &= 0 - 0 + b(0) + \frac{1}{2}\left(\sigma^2 - \sigma^2\right) \\ &= 0 \end{aligned}$$

$$E\left\{V_i^2\right\} = E\left\{\left(R_n v_n - R_i v_i + b(v_i - v_n) + \tilde{V}_i\right)^2\right\}$$

where $\tilde{V}_i = \frac{1}{2}\left(v_i^2 - v_n^2\right)$

$$E\left\{V_i^2\right\} = R_n^2 v_n^2 - 2R_n R_i v_n v_i + 2bR_n v_n (v_i - v_n) + 2R_n v_n \tilde{V}_i + R_i^2 v_i^2$$

$$- 2bR_i v_i (v_i - v_n) - 2R_i v_i \tilde{V}_i + b^2 (v_i - v_n)^2$$

$$+ 2b\tilde{V}_i (v_i - v_n) + \tilde{V}_i^2$$

$$E\left\{V_i^2\right\} = R_n^2 E\left\{v_n^2\right\} - 2R_n R_i E\{v_n v_i\} + 2bR_n E\left\{v_n v_i - v_n^2\right\}$$

$$+ 2R_n E\left\{v_n \tilde{V}_i\right\} + R_i^2 E\left\{v_i^2\right\} - 2bR_i E\left\{v_i^2 - v_n v_i\right\}$$

$$- 2R_i E\left\{v_i \tilde{V}_i\right\} + b^2 E\left\{(v_i - v_n)^2\right\} + 2bE\left\{\tilde{V}_i v_i - \tilde{V}_i v_n\right\}$$

$$+ E\left\{\tilde{V}_i^2\right\}$$

$$E\left\{v_n v_i - v_n^2\right\} = E\{v_n v_i\} - E\left\{v_n^2\right\} = -\sigma^2$$

$$E\left\{v_i^2 - v_n v_i\right\} = -E\left\{v_n v_i - v_n^2\right\} = \sigma^2$$

$$E\{v_n \tilde{V}_i\} = \frac{1}{2} E\left\{v_n v_i^2 - v_n^3\right\} = \frac{1}{2}\left(E\{v_n\} E\left\{v_i^2\right\} - E\left\{v_n^3\right\}\right) = 0$$

$$E\left\{v_i \tilde{V}_i\right\} = \frac{1}{2} E\left\{-v_i v_n^2 + v_i^3\right\} = \frac{1}{2}\left(-E\{v_i\} E\left\{v_n^2\right\} + E\left\{v_i^3\right\}\right) = 0$$

$$b^2E\left\{(v_i - v_n)^2\right\} = b^2\left(E\left\{v_i^2\right\} - 2E\{v_i v_n\} + E\left\{v_n^2\right\}\right) = 2b^2\sigma^2$$

$$\begin{aligned}
E\left\{\tilde{V}_i^2\right\} &= \frac{1}{4}E\left\{(v_i^2 - v_n^2)^2\right\} \\
&= \frac{1}{4}E\left\{v_i^4 - 2v_i^2 v_n^2 + v_n^4\right\} \\
&= \frac{1}{4}\left(E\left\{v_i^4\right\} - 2E\left\{v_i^2\right\}E\left\{v_n^2\right\} + E\left\{v_n^4\right\}\right) \\
&= \frac{1}{4}\left(3\sigma^4 - 2\sigma^4 + 3\sigma^4\right) \\
&= \sigma^4
\end{aligned}$$

Thus,

$$\begin{aligned}
E\left\{V_i^2\right\} &= R_n^2\sigma^2 - 2bR_n\sigma^2 + R_i^2\sigma^2 - 2bR_i\sigma^2 + 2b^2\sigma^2 + \sigma^4 \\
&= \sigma^4 + \sigma^2\left(R_i^2 + R_n^2 - 2b(R_n + R_i) + 2b^2\right) \qquad (3.10) \\
&= \sigma^4 + \sigma^2\left((R_n - b)^2 + (R_i - b)^2\right)
\end{aligned}$$

In a similar manner,

$$E\{V_i V_j\}_{i \neq j} = \frac{\sigma^4}{2} + \sigma^2(R_n - b)^2 \qquad (3.11)$$

From the results obtained in Eq. (3.10) and Eq. (3.11), the covariance matrix for the noise vector $\vec{V}$ is explicitly given by:

$$R = \sigma^2 \begin{bmatrix} d_1 & c & c & \bullet & \bullet & \bullet & c \\ c & d_2 & c & & & & c \\ c & c & d_3 & c & & & c \\ c & c & c & d_4 & & & c \\ \bullet & & & & \bullet & & \bullet \\ \bullet & & & & & \bullet & c \\ c & c & \bullet & \bullet & \bullet & c & d_{n-1} \end{bmatrix}_{(n-1)\times(n-1)} \tag{3.12}$$

where

$$c = \frac{\sigma^2}{2} + (R_n - b)^2$$

and

$$d_i = \sigma^2 + (R_n - b)^2 + (R_i - b)^2 \quad \text{for } i = 1, 2, 3, \ldots, n-1$$

The closed-form solution presented in this thesis requires the inverse of **R**. It is desirable to find a closed-form solution for this inverse to reduce the computation load of our GPS positioning algorithm. If **R** is redefined as, $R = c\sigma^2\tilde{R}$, then:

$$R^{-1} = \frac{1}{c\sigma^2}\tilde{R}^{-1} \tag{3.13}$$

where:

$$\tilde{R} = \begin{bmatrix} \frac{d_1}{c} & 1 & 1 & \bullet & 1 \\ 1 & \frac{d_2}{c} & 1 & \bullet & 1 \\ 1 & 1 & \frac{d_3}{c} & \bullet & 1 \\ \bullet & & & \bullet & 1 \\ 1 & 1 & \bullet & \bullet & \frac{d_{n-1}}{c} \end{bmatrix} \tag{3.14}$$

The elements of the diagonal of $\tilde{R}$ are a function of b, the clock bias error, and σ, the standard deviation of the measurement noise. For implementation purposes it is desirable to remove this dependency before finding a solution for $\tilde{R}^{-1}$.

The diagonal elements of $\tilde{R}$ are given by:

$$\frac{d_i}{c} = \frac{\sigma^2 + (R_n - b)^2}{\frac{\sigma^2}{2} + (R_n - b)^2} + \frac{(R_i - b)^2}{\frac{\sigma^2}{2} + (R_n - b)^2} \tag{3.15}$$

Since $\sigma^2 \ll (R_n - b)^2$, Eq. (3.15) can be simplified as follows:

$$\frac{d_i}{c} \approx 1 + \frac{(R_i - b)^2}{(R_n - b)^2} = \tilde{d}_i \tag{3.16}$$

For most positioning applications $b \ll (R_i - b)^2$ and Eq. (3.16) can be simplified further as shown in Eq. (3.17). To further strengthen the validity of the assumptions made to form Eq. (3.16) and Eq. (3.17), R_n can be picked as the largest of all available pseudoranges. Eq. (3.17) defines $\tilde{d}_i$ for the rest of this thesis:

$$\frac{d_i}{c} \approx 1 + \frac{R_i^2}{R_n^2} = \tilde{d}_i \qquad (3.17)$$

Thus,

$$\tilde{R} \approx \begin{bmatrix} \tilde{d}_1 & 1 & 1 & \bullet & 1 \\ 1 & \tilde{d}_2 & 1 & \bullet & 1 \\ 1 & 1 & \tilde{d}_3 & \bullet & 1 \\ \bullet & & & \bullet & 1 \\ 1 & 1 & \bullet & \bullet & \tilde{d}_{n-1} \end{bmatrix} \qquad (3.18)$$

To find $\tilde{R}^{-1}$ let's define the diagonal matrix

$$D = \mathrm{Diag}\left(\begin{bmatrix} \tilde{d}_1 - 1 \\ \tilde{d}_2 - 1 \\ \bullet \\ \bullet \\ \bullet \\ \tilde{d}_{n-1} - 1 \end{bmatrix}\right), \text{ and the vector } \vec{e} = \begin{bmatrix} 1 \\ 1 \\ \bullet \\ \bullet \\ \bullet \\ 1 \end{bmatrix}_{(n-1) \times 1}$$

Obviously,

$$\tilde{R} = D + \vec{e}\vec{e}^T = D - \vec{e}(-1)^{-1}\vec{e}^T$$

Applying the Matrix Inversion Lemma,

$$\tilde{R}^{-1} = D^{-1} + D^{-1}\vec{e}\left(-1 - \vec{e}^T D^{-1}\vec{e}\right)^{-1}\vec{e}^T D^{-1}$$

$$= D^{-1} - \frac{1}{1 + \vec{e}^T D^{-1}\vec{e}} D^{-1}\vec{e}\vec{e}^T D^{-1}$$

If we define $D^{-1} = \text{Diag}\left\{\begin{bmatrix} r_1 \\ r_2 \\ \bullet \\ \bullet \\ \bullet \\ r_{n-1} \end{bmatrix}\right\}_{(n-1)\times(n-1)}$, i.e., $r_i = \frac{1}{\tilde{d}_i - 1}$

then, $D^{-1}\vec{e} = \begin{bmatrix} r_1 \\ r_2 \\ \bullet \\ \bullet \\ \bullet \\ r_{n-1} \end{bmatrix}$, $\vec{e}^T D^{-1}\vec{e} = \sum_{k=1}^{n-1} r_k$, and $\left(D^{-1}\vec{e}\vec{e}^T D^{-1}\right)_{i,j} = r_i r_j$.

Thus,

$$\left(\tilde{R}^{-1}\right)_{i,i} = \frac{1}{\sigma^2 c}\left(r_i - \frac{r_i^2}{1 + \sum_{k=1}^{n-1} r_k}\right), \; i = 1, 2, \ldots, n-1$$
$$\left(\tilde{R}^{-1}\right)_{i,j} = \frac{1}{\sigma^2 c}\left(-\frac{r_i r_j}{1 + \sum_{k=1}^{n-1} r_k}\right), \; i \neq j \tag{3.19}$$

3.1.4 Minimum Variance Estimate Solution.

Using the linear regression from Eq. (3.6) as a starting point, the aim is to obtain an estimate $\hat{\vec{u}}$. The $\hat{\vec{u}}$ that minimizes the estimation error as weighted by the inverse covariance of the noise must be obtained. Recognizing that the equation error, also known as the return difference or measurement residual, is $(\vec{Z} - H\vec{u})$, the estimation problem can be formulated as follows:

$$\min_{\vec{u}}\left[\left(\vec{Z} - H\vec{u}\right)^T R^{-1}\left(\vec{Z} - H\vec{u}\right)\right] \tag{3.20}$$

Eq. (3.20) can be expanded to obtain:

$$\vec{Z}^T R^{-1}\vec{Z} + \vec{u}^T H^T R^{-1} H\vec{u} - 2\vec{u}^T H^T R^{-1}\vec{Z} \tag{3.21}$$

Since a minimization over $\vec{u}$ is needed, Eq. (3.21) is differentiated with respect to $\vec{u}$ and set equal to zero yielding the following expression:

$$0 + 2H^T R^{-1} H\vec{u} - 2H^T R^{-1}\vec{Z} = 0 \tag{3.22}$$

Rearranging the expression and solving for $\vec{u}$ produces the desired solution:

$$\hat{\vec{u}} = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1}\vec{Z} \tag{3.23}$$

In order to demonstrate that the stationary point solution in Eq. (3.23) is indeed a minimum, the hessian matrix of Eq. (3.21) must be verified. The resulting hessian matrix is $(H^T R^{-1} H)$, which by definition is always positive definite, providing the necessary and sufficient conditions for minimization. Furthermore, since the existence of the hessian inverse in Eq. (3.23) is guaranteed, the existence of a solution is also guaranteed.

Eq. (3.23) is a *closed-form* solution to the GPS pseudorange equations. However, this solution is dependent on σ, the pseudorange measurement noise standard deviation. To simplify the solution for implementation, it is noted that Eq. (3.13) shows the noise covariance R^{-1} as simply $\tilde{R}^{-1}$ premultiplied by a scalar quantity. In Eq. (3.23) the scalar premultiplier of R^{-1} will cancel out; therefore the estimation solution shown in Eq. (3.23) can be rewritten in a equivalent form as:

$$\hat{\vec{u}} = \left(H^T \tilde{R}^{-1} H\right)^{-1} H^T \tilde{R}^{-1} \vec{Z} \tag{3.24}$$

Eq. (3.24) is used for coding the experimental Matlab algorithm. It must be noted that there are no big matrix inversions associated with this solution since $\tilde{R}^{-1}$ has been determined analytically and can be coded directly into the algorithm. The only inversion that needs to be performed is that of the small 4×4 matrix $(H^T \tilde{R}^{-1} H)$, which can be hardwired into the receiver's algorithm.

3.1.5 Estimate Error Covariance.

It follows from Eq. (3.20) and (3.23) that the covariance of the estimate error is given by:

$$P_{\vec{u}} \equiv E\left\{\left(\vec{u} - \hat{\vec{u}}\right)\left(\vec{u} - \hat{\vec{u}}\right)^T\right\} \tag{3.25}$$

If we expand $\vec{u} - \hat{\vec{u}}$,

$$
\begin{aligned}
\vec{u} - \hat{\vec{u}} &= \vec{u} - (H^T\tilde{R}^{-1}H)^{-1}H^T\tilde{R}^{-1}\vec{Z} \\
&= \vec{u} - \left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\left(H\vec{u} + \vec{V}\right) \\
&= \vec{u} - \left(H^T\tilde{R}^{-1}H\right)^{-1}\left(H^T\tilde{R}^{-1}H\right)\vec{u} - \left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{V} \\
&= \vec{u} - \vec{u} - \left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{V} \\
&= -\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{V}
\end{aligned}
$$

$$
\begin{aligned}
\left(\vec{u} - \hat{\vec{u}}\right)\left(\vec{u} - \hat{\vec{u}}\right)^T &= \left(\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{V}\right)\left(\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{V}\right)^T \\
&= \left(\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{V}\right)\left(\left(H^T\tilde{R}^{-1}\vec{V}\right)^T\left(H^T\tilde{R}^{-1}H\right)^{-1}\right) \\
&= \left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{V}\vec{V}^T\tilde{R}^{-1}H\left(H^T\tilde{R}^{-1}H\right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
E\left\{\left(\vec{u} - \hat{\vec{u}}\right)\left(\vec{u} - \hat{\vec{u}}\right)^T\right\} &= \left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}E\left\{\vec{V}\vec{V}^T\right\}\tilde{R}^{-1}H\left(H^T\tilde{R}^{-1}H\right)^{-1} \\
&= \left(H^TR*^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\sigma^2 cc\tilde{R}\tilde{R}^{-1}H\left(H^T\tilde{R}^{-1}H\right)^{-1} \\
&= \sigma^2 c\left(H^T\tilde{R}^{-1}H\right)^{-1}\left(H^T\tilde{R}^{-1}H\right)\left(H^T\tilde{R}^{-1}H\right)^{-1} \\
&= \sigma^2 c\left(H^T\tilde{R}^{-1}H\right)
\end{aligned}
$$

$$P_{\hat{u}} = \sigma^2 c\left(H^T\tilde{R}^{-1}H\right)^{-1} \tag{3.26}$$

Unlike the solution estimate, the covariance $P_{\hat{u}}$ is dependent on σ; hence, σ must be known or estimated, in order to estimate the error covariance.

Substituting Eq. (3.6) into Eq. (3.24) yields:

$$\begin{aligned}
\hat{\vec{u}} &= \left(H^T\tilde{R}^{-1}H\right)^{-1} H^T\tilde{R}^{-1}\left(H\vec{u} + \vec{V}\right) \\
&= \left(H^T\tilde{R}^{-1}H\right)^{-1}\left(H^T\tilde{R}^{-1}H\right)\vec{u} + \left(H^T\tilde{R}^{-1}H\right)^{-1} H^T\tilde{R}^{-1}\vec{V} \\
&= \vec{u} + \left(H^T\tilde{R}^{-1}H\right)^{-1} H^T\tilde{R}^{-1}\vec{V}
\end{aligned} \tag{3.27}$$

Now let's substitute Eq. (3.27) into the return difference equation:

$$\begin{aligned}
\tilde{Z} = Z - H\hat{\vec{u}} &= H\vec{u} + \vec{V} - H\left(\vec{u} + \left(H^T\tilde{R}^{-1}H\right)^{-1} H^T\tilde{R}^{-1}\vec{V}\right) \\
&= H\vec{u} + \vec{V} - H\vec{u} - H\left(H^T\tilde{R}^{-1}H\right)^{-1} H^T\tilde{R}^{-1}\vec{V} \\
&= \vec{V} - H\left(H^T\tilde{R}^{-1}H\right)^{-1} H^T\tilde{R}^{-1}\vec{V} \\
&= \left(I_{n-1} - H\left(H^T\tilde{R}^{-1}H\right)^{-1} H^T\tilde{R}^{-1}\right)\vec{V}
\end{aligned}$$

If we define $M = I_{n-1} - H\left(H^T\tilde{R}^{-1}H\right)^{-1} H^T\tilde{R}^{-1}$, then the return difference can be expressed as:

$$\tilde{Z} = M\vec{V} \tag{3.28}$$

Claim: The matrix **M** is indempotent, viz., $M^2 = M$

Proof:

$$M^2 = \left(I_{n-1} - H\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\right)\left(I_{n-1} - H\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\right)$$

$$= I_{n-1} - 2H\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1} + \left(H\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\right)^2$$

$$= I_{n-1} - 2H\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1} + H\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}$$

$$= I_{n-1} - H\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}$$

$$= M$$

Let's now define the weighted return difference:

$$\tilde{\mathbf{z}} = R_e^{-1}\tilde{Z} \tag{3.29}$$

Where R_e^{-1} is obtained from the Cholesky decomposition of $\tilde{R}$, viz., $\tilde{R} = R_e R_e^T$. We now calculate the scalar quantity:

$$\tilde{\mathbf{z}}^T\tilde{\mathbf{z}} = \tilde{Z}^T\left(R_e^{-1}\right)^T R_e^{-1}\tilde{Z} = \tilde{Z}^T\left(R_e^T\right)^{-1}R_e^{-1}\tilde{Z} = \tilde{Z}^T\left(R_e R_e^T\right)^{-1}\tilde{Z}$$

$$= \tilde{Z}^T\tilde{R}^{-1}\tilde{Z}$$

$$= \vec{V}^T M^T\tilde{R}^{-1}M\vec{V}$$

Since $\tilde{\mathbf{z}}^T\tilde{\mathbf{z}}$ is a scalar, then:

$$E\left\{\tilde{z}^T\tilde{z}\right\} = E\left\{\vec{V}^T M^T \tilde{R}^{-1} M\vec{V}\right\} = E\left\{Tr\left(\vec{V}^T M^T \tilde{R}^{-1} M\vec{V}\vec{V}^T\right)\right\}$$

$$= E\left\{Tr\left(M^T \tilde{R}^{-1} M\vec{V}\vec{V}^T\right)\right\} = Tr\left(M^T \tilde{R}^{-1} ME\left\{\vec{V}\vec{V}^T\right\}\right)$$

$$= Tr\left(M^T \tilde{R}^{-1} Mc\sigma^2 \tilde{R}\right) = c\sigma^2 Tr\left(M^T \tilde{R}^{-1} M\tilde{R}\right)$$

<u>Claim:</u> $\tilde{R}^{-1}M\tilde{R} = M^T$

Proof:

$$\tilde{R}^{-1}M\tilde{R} = \tilde{R}^{-1}(I_{n-1} - H\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1})\tilde{R}$$

$$= I_{n-1} - \tilde{R}^{-1}H\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T$$

$$= \left(I_{n-1} - H\left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\right)^T$$

$$= M^T$$

Hence, $M^T\tilde{R}^{-1}M\tilde{R} = M^TM^T$ and:

$$E\left\{\tilde{z}^T\tilde{z}\right\} = c\sigma^2 Tr\left(M^TM^T\right)$$

$$= c\sigma^2 Tr\left((MM)^T\right)$$

$$= c\sigma^2 Tr\left(M^2\right)$$

$$= c\sigma^2 Tr(M)$$

Now, we calculate:

$$\begin{aligned}
Tr(M) &= Tr\left(I_{n-1} - H\left(H^T\tilde{R}^{-1}H\right)^{-1} H^T\tilde{R}^{-1}\right) \\
&= Tr(I_{n-1}) - Tr\left(H\left(H^T\tilde{R}^{-1}H\right)^{-1} H^T R^{*-1}\right) \\
&= (n-1) - Tr\left(\left(H^T\tilde{R}^{-1}H\right)^{-1} H^T\tilde{R}^{-1}H\right) \\
&= (n-1) - Tr(I_4) \\
&= n-5
\end{aligned}$$

Thus,

$$E\left\{\tilde{z}^T\tilde{z}\right\} = c\sigma^2(n-5) = \left(\frac{\sigma^4}{2} + \sigma^2(R_n - b)\right)(n-5) \tag{3.30}$$

Given that $E\{\tilde{z}^T\tilde{z}\}$ is a scalar and $E\{\tilde{z}^T\tilde{z}\} = \tilde{Z}^T\tilde{R}^{-1}\tilde{Z}$ we can solve for σ^2 by rearranging Eq. (3.30) as follows:

$$\frac{\sigma^4}{2} + \sigma^2(R_n - b) - \frac{\tilde{Z}^T\tilde{R}^{-1}\tilde{Z}}{n-5} = 0 \tag{3.31}$$

Using the quadratic equation and taking only the positive root of Eq. (3.31) an estimate of σ^2 is given by:

$$\sigma^2 \approx -(R_n - \hat{b}) + \sqrt{(R_n - \hat{b})^2 + \frac{2(\tilde{Z}^T\tilde{R}^{-1}\tilde{Z})}{n-5}} \tag{3.32}$$

In conclusion, the derived linear regression (3.6), which consists of $(n-1)$ equations, requires that $n-1$ be at least four to provide an initial estimate of the four parameters in $\vec{u}$. This implies that a minimum satellite availability of five is required to produce the solution given in Eq. (3.24). An extra satellite is required for the prediction of the estimation error covariance $P_{\hat{u}}$. Thus, a minimum satellite availability of six is required to produce an initial estimate of the four parameters in $\hat{u}$ and the estimation error covariance

3.2 *Kalman Update Solution*

This section presents the development of the *Kalman update* GPS position determination algorithm, where the pseudorange equation which was subtracted from the first $n - 1$ equations is finally used. The *Kalman update* step is a supplementary process that improves on the *closed-form* solution presented in Section 3.1. The concept behind the *Kalman update* is discussed followed by the complete mathematical derivation of the solution.

3.2.1 *Kalman Update Concept*

The concept behind the *Kalman update* solution approach is similar to that of a conventional Kalman Filter. The *closed-form* solution in Section 3.1 provides a preliminary GPS solution estimate ($\hat{u}$) and the associated error covariance matrix ($P_{\hat{u}}$). Recalling that this solution was produced without making use of the nonlinear n^{th} pseudorange equation in Eq. (3.5), we can perceive this n^{th} equation as a new measurement which can be used to update the previous estimate the same way that it would be accomplished during the update cycle of an extended Kalman Filter. The approach that is used begins with the linearization of Eq. (3.5) about a nominal position estimate. The linearized equation is then manipulated into the standard linear measurement form as described in [3], and used to update the estimate. Since the measurement Eq. (3.5) is nonlinear, it may be necessary for the process to continue in an iterative manner until convergence within a predefined tolerance. Simulation results show that the algorithm converges after two to three iterations.

The Kalman update algorithm that is presented in this section differs from the basic Kalman Filter developed by Kalman ([7], [8]) in that the measurement that is used to update the preliminary estimate is correlated with the preliminary estimate. The conventional Kalman Filter update equation does not allow for correlation between the new measurement and the previous estimate; hence, a novel Kalman-like update equation that accommodates this correlation and that addresses the specific measurement situation on hand, needs to be derived.

3.2.2 Linearized Measurement.

The first step in the mathematical development of the *Kalman update* algorithm is to linearize Eq. (3.4) about a nominal user position (u_{xo}, u_{yo}, u_{zo}) by performing a Taylor series expansion and neglecting second and higher order terms. The linearized equation obtained is given by:

$$
\begin{aligned}
R_n \approx\ & \frac{(u_{xo} - x_n)}{\sqrt{(u_{xo} - x_n)^2 + (u_{yo} - y_n)^2 + (u_{zo} - z_n)^2}}(u_x - u_{xo}) \\
& + \frac{(u_{yo} - y_n)}{\sqrt{(u_{xo} - x_n)^2 + (u_{yo} - y_n)^2 + (u_{zo} - z_n)^2}}(u_y - u_{yo}) \\
& + \frac{(u_{zo} - z_n)}{\sqrt{(u_{xo} - x_n)^2 + (u_{yo} - y_n)^2 + (u_{zo} - z_n)^2}}(u_z - u_{zo}) \\
& + \sqrt{(u_{xo} - x_n)^2 + (u_{yo} - y_n)^2 + (u_{zo} - z_n)^2} + b + v_n
\end{aligned}
$$

By defining the regressor *h* for this scalar measurement equation as follows:

$$h \equiv \begin{bmatrix} \dfrac{(u_{xo} - x_n)}{\sqrt{(u_{xo} - x_n)^2 + (u_{yo} - y_n)^2 + (u_{zo} - z_n)^2}} \\ \dfrac{(u_{yo} - y_n)}{\sqrt{(u_{xo} - x_n)^2 + (u_{yo} - y_n)^2 + (u_{zo} - z_n)^2}} \\ \dfrac{(u_{zo} - z_n)}{\sqrt{(u_{xo} - x_n)^2 + (u_{yo} - y_n)^2 + (u_{zo} - z_n)^2}} \\ 1 \end{bmatrix} \tag{3.33}$$

the linearized equation can be rewritten as:

$$R_n \approx h^T \vec{u} + \omega_v - \frac{(u_{xo} - x_n)u_{xo} + (u_{yo} - y_n)u_{yo} + (u_{zo} - z_n)u_{zo}}{\sqrt{(u_{xo} - x_n)^2 + (u_{yo} - y_n)^2 + (u_{zo} - z_n)^2}} + \sqrt{(u_{xo} - x_n)^2 + (u_{yo} - y_n)^2 + (u_{zo} - z_n)^2}$$

The goal is to reduce the above equation into the form of a linear measurement model described by:

$$Z_n = h^T \vec{u} + v_n \tag{3.34}$$

In order to achieve this goal, Z_n must be defined as:

$$Z_n \equiv R_n - \sqrt{(u_{xo} - x_n)^2 + (u_{yo} - y_n)^2 + (u_{zo} - z_n)^2} + \frac{(u_{xo} - x_n)u_{xo} + (u_{yo} - y_n)u_{yo} + (u_{zo} - z_n)u_{zo}}{\sqrt{(u_{xo} - x_n)^2 + (u_{yo} - y_n)^2 + (u_{zo} - z_n)^2}}$$

which can be simplified into the following:

$$Z_n = R_n + \frac{(u_{xo} - x_n)x_n + (u_{yo} - y_n)y_n + (u_{zo} - z_n)z_n}{\sqrt{(u_{xo} - x_n)^2 + (u_{yo} - y_n)^2 + (u_{zo} - z_n)^2}} \tag{3.35}$$

Now that the n^{th} pseudorange measurement equation is approximated into the appropriate linear measurement model form defined in Eq. (3.34), it can be used to update the solution obtained from the *closed-form* algorithm using a Kalman type update approach. Using a linear measurement model simplifies the solution by allowing the use of linear Kalman filtering techniques, as opposed to using an Extended Kalman Filter or increasing the order of the filter to accommodate a nonlinear measurement equation. Keeping in mind that Z_n is actually part of the measurements that were used to obtain the *closed-form* solution and not a new measurement as would be the case in a conventional Kalman Filter application, hence the new measurement and the previous estimate are correlated. This is a violation to the basic assumptions used in the derivation of the conventional Kalman Filter update equations. A Kalman-like update equation that can accommodate correlation between the new measurement and the previous estimate needs to be derived.

3.2.3 Noise Statistics

In order to derive the new Kalman-like update equation, it is necessary to know the relationship between the noise in the new measurement (v_n) and the previous estimate being the solution obtained from the *closed-form* algorithm. The linear regression used for the *closed-form* algorithm was defined in Eq. (3.6) as $\vec{Z} = H\vec{u} + \vec{V}$, and the statistics of the noise vector $\vec{V}$ were derived in Section 3.1.3. The *closed-form* algorithm produced an estimate of the GPS unknown parameters, $\hat{\vec{u}}$, defined in Eq. (3.24), and an estimate of its covariance matrix ($P_{\hat{\vec{u}}}$), defined in Eq. (3.26). Using the knowledge of the estimated GPS solution, the true GPS parameter vector can be defined as,

$$\vec{u} = \hat{\vec{u}} + \vec{W} \tag{3.36}$$

where $\vec{W} \sim N(0, P_{\hat{\vec{u}}})$. The correlation of interest between v_n and $\vec{W}$ can be defined as:

$$p \equiv E\left\{\vec{W}v_n\right\} = E\left\{v_n\vec{W}\right\} \tag{3.37}$$

To determine the relationship between $\vec{W}$ and v_n, the linear regression in Eq. (3.6) is multiplied from the left by $H^T\tilde{R}^{-1}$ yielding the expression:

$$H^T\tilde{R}^{-1}\vec{Z} = H^T\tilde{R}^{-1}H\vec{u} + H^T\tilde{R}^{-1}\vec{V} \tag{3.38}$$

Eq. (3.38) can be solved for $\vec{u}$ to obtain:

$$\vec{u} = \left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{Z} - \left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{V} \tag{3.39}$$

The first term on the right hand side of Eq. (3.39) is recognized from Eq. (3.24) as $\hat{\vec{u}}$; therefore, an expression for $\vec{W}$ in terms of $\vec{V}$ is obtained.

$$\vec{W} = \left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{V} \tag{3.40}$$

Next the relationship between $\vec{V}$ and v_n is determined by exploiting the noise statistics derived in Section 3.1.3.

$$E\{V_i v_n\} = E\left\{R_n v_n^2 - R_i v_i v_n + b v_i v_n - b v_n^2 + \frac{1}{2}\left(v_i^2 v_n - v_n^3\right)\right\}$$
$$= \sigma^2 R_n - \sigma^2 b$$
$$= \sigma^2 (R_n - b) \tag{3.41}$$

Eq. (3.41), which represents the variance between any single element of $\vec{V}$ and v_n, can be generalized to obtain the following covariance matrix:

$$E\left\{\vec{V} v_n\right\} = \sigma^2 (R_n - b) \begin{bmatrix} 1 \\ 1 \\ \bullet \\ \bullet \\ \bullet \\ 1 \end{bmatrix}_{((n-1) \times 1)} \tag{3.42}$$

Using Eq. (3.40) and (3.42), an expression for the covariance between $\vec{W}$ and v_n is determined:

$$E\left\{\vec{W} v_n\right\} = \left(H^T \tilde{R}^{-1} H\right)^{-1} H^T \tilde{R}^{-1} E\left\{\vec{V} v_n\right\}$$
$$= \sigma^2 (R_n - b) \left(H^T \tilde{R}^{-1} H\right)^{-1} H^T \tilde{R}^{-1} \begin{bmatrix} 1 \\ 1 \\ \bullet \\ \bullet \\ \bullet \\ 1 \end{bmatrix}_{((n-1) \times 1)} \tag{3.43}$$

The goal is to derive a Kalman-like update equation to refine the unknown GPS parameters vector estimate, $\hat{\vec{u}}$, and its covariance matrix, $(P_{\hat{u}})$, both produced by the *closed-form* algorithm developed in Section 3.1. Towards this end, an augmented linear regression is formulated by combining Eq. (3.34) and (3.36). The augmented linear regression is expressed as:

$$\vec{Z}_a = H_a \vec{u} + \vec{V}_a \tag{3.44}$$

where $\vec{Z}_a$ is the 5×1 augmented "measurement" vector defined as:

$$\vec{Z}_a \equiv \begin{bmatrix} \hat{\vec{u}} \\ Z_n \end{bmatrix}$$

H_a is the 5×4 augmented regressor defined as:

$$H_a \equiv \begin{bmatrix} I \\ h^T \end{bmatrix}$$

and $\vec{V}_a$ is the 5×1 augmented "measurement noise" vector defined as:

$$\vec{V}_a \equiv \begin{bmatrix} \vec{W} \\ v_n \end{bmatrix}$$

In the derivation that follows, to distinguish the preliminary estimate $\hat{\vec{u}}$ and $P_{\vec{u}}$ as produced by the *closed-form* algorithm from the estimate that will be obtained through the Kalman update, the following notation is used:

$\hat{\vec{u}}^-$ and $P_{\vec{u}}^-$ represent the estimate and the estimation error covariance prior to the update.

$\hat{\vec{u}}^+$ and $P_{\vec{u}}^+$ represents the estimate and the estimation error covariance following the update.

In order to obtain the updated estimates from the augmented linear regression in (3.44), it is necessary to derive the covariance of the augmented noise vector $\vec{V}_a$. Since the statistics of the noise components in $\vec{V}_a$ have already being determined, the equation error covariance matrix, R_a, is given by:

$$R_a = \begin{bmatrix} P_{\vec{u}}^{-} & p \\ p^T & \sigma^2 \end{bmatrix} \tag{3.45}$$

The updated GPS minimum variance solution estimate and the associated covariance are then given by the expressions:

$$\hat{\vec{u}}^{+} = P_{\vec{u}}^{+} H_a^T R_a^{-1} \vec{Z}_a \tag{3.46}$$

$$P_{\vec{u}}^{+} = \left(H_a^T R_a^{-1} H_a \right)^{-1} \tag{3.47}$$

The expressions in Eq. (3.46) and (3.47) are sufficient to obtain the required updates, but it is desirable to manipulate and reduce the equations into the more familiar and computationally efficient form of the classical Kalman filter update equations. After lengthy manipulations and applying the Matrix Inversion Lemma, the Kalman-like update equations in the desired form are obtained, viz.,

$$\hat{\vec{u}}^{+} = \hat{\vec{u}}^{-} + K \left(Z_n - h^T \hat{\vec{u}}^{-} \right) \tag{3.48}$$

$$P_{\vec{u}}^{+} = \left\{ I - \left[\left(1 - p^T h \right) K + p \right] h^T \right\} Y \tag{3.49}$$

where the intermediate variable *Y* is the modified preupdate covariance matrix given by

$$Y = P_{\hat{u}}^{-} + \frac{h^{T}P_{\hat{u}}^{-}h - 1}{\left(1 - p^{T}h\right)^{2}} pp^{T} + \frac{1}{1 - p^{T}h}\left(P_{\hat{u}}^{-}hp^{T} + ph^{T}P_{\hat{u}}^{-}\right) \tag{3.50}$$

and *K* is the modified Kalman filter gain given by:

$$K = \frac{1}{1 - p^{T}h}\left(\frac{1}{1 + h^{T}Yh}Yh - p\right) \tag{3.51}$$

The parameter estimate update, Eq. (3.48), appears identical to that of the classical Kalman Filter update equations. However, this is not the case since the Kalman Filter gain, Eq. (3.51), is not the same.

A quick verification of the derived update equations is carried out to confirm the validity of the new equations. Note that in the special case of the classical Kalman Filter with no correlation, $p = 0$ and $Y = P_{\hat{u}}^{-}$. For this special case the classical Kalman Filter update formulae are indeed recovered:

$$K = \frac{1}{1 + h^{T}P_{\hat{u}}^{-}h}P_{\hat{u}}^{-}h$$

$$\hat{u}^{+} = \hat{u}^{-} + K\left(Z_{n} - h^{T}\hat{u}^{-}\right)$$

$$P_{\hat{u}}^{+} = \left(I - Kh^{T}\right)P_{\hat{u}}^{-}$$

Eq. (3.48) to (3.49) are used in the Matlab implementation of the *Kalman update* algorithm. The *Kalman update* algorithm is intended to refine the GPS *closed-form* solution estimate in a direct and non-recursive manner. However, since the measurement Eq. (3.5) is nonlinear, it may be necessary for the process to continue in an iterative manner until convergence within a predefined tolerance. Recalling that the new measurement used by the *Kalman update* algorithm is actually the n^{th} pseudorange equation in Eq. (3.5) which has been linearized about the position estimate produced by the *closed-form* algorithm, implies that how well the linearization fits the true unknown GPS parameters is dependent on how good the solution produced by the close-form algorithm is to begin with. In order to alleviate this undesired dependency, after the *Kalman Update* algorithm has been applied once, and produces an improved solution estimate, Eq. (3.5) is once again linearized about the improved position estimate (note that the user clock bias plays no roll here) producing a *new* linear measurement equation. This is akin to the iterated Kalman Filter algorithm used in Extended Kalman Filtering. The *Kalman update* algorithm is applied a second time using the preliminary estimate and estimation error covariance available prior to the update and produced by the linear *closed-form* algorithm, <u>not</u> the solution obtained as a result of the previous application of the *Kalman update*. Theoretically, this process can be continued recursively until convergence to the best possible solution is achieved; however, it was found experimentally that after 2 or 3 application the change in the solution estimate is insignificant. Hence, the algorithm is hardwired to perform three iterations. As such, the algorithm is "not iterative."

4. Experimental Results and Analysis

This chapter presents the experimental portion of the thesis. The first part discusses how the experiment was set up and how the Monte Carlo trials were run. The experimental results are then presented and the chapter sums up with a detailed analysis of the results.

4.1 Experimental Setup

The *closed-form*, linear regression algorithm developed in this paper requires at least six pseudorange measurements to produce a stand-alone GPS solution and a prediction of the position estimation error covariance. In terms of satellite availability, the worst case scenario occurs at latitudes in the range of 35 to 55 degrees where there are at most six satellites available 20 percent of the time. However, most of the time, more than six satellites are in view. The novel algorithm uses all n available pseudorange measurements to produce the GPS solution. Satellite availability is not dependent on user position longitude; hence, a fixed user position in the 35 to 55 degree latitude range over the continental Unites States, 40° N latitude, 105° W longitude, at an altitude of 300 m was selected. The geographic coordinates are converted to ECEF coordinates and used to generate the experimental data sets using GPSoft's Satellite Navigation Toolbox for Matlab [24]. The experimental data sets were generated for 12 scenarios which showed greatest diversity in satellite availability and geometry.

The Satellite Navigation Toolbox is used to generate realistic GPS satellite position data from which true ranges can be calculated between all in view GPS satellites and the position of the selected receiver. After adding an arbitrary clock bias of 1000 m to all the ranges, a zero mean random noise of preselected standard deviation $\sigma = 100$ meters, is superimposed to represent the Gaussian measurement noise. The Satellite Navigation Toolbox has the capability of simulating realistic noise corrupted pseudorange measurements which could be applied directly to the GPS position determination algorithm as it would be the case in a real world scenario. The approach used in our experiments, simulating just the GPS satellite ephemeris data and producing the simulated pseudoranges, was preferable for the following reasons:

- It proves a more structured data set for analysis of the algorithms since only the desired effects are being considered and the amount of noise corruption on the pseudorange measurements is exactly controlled; and,
- Since the pseudoranges are produced starting from exactly known position coordinates, comparisons against the true position for determining the algorithm's accuracy are possible.

In addition to producing experimental results using the novel two step algorithm developed in this paper, results were also produced using the conventional ILS algorithm to provide a comparison baseline. The ILS algorithm is commonly implemented using the "best four" satellites. The best four satellites are the four satellites in view whose pseudoranges form the regressor matrix H with the lowest condition number [18], [25]. However, the results in [1] show the ILS algorithm performing better when the pseudoranges from all satellites in view are used. Consequently, in the simulation, the ILS algorithm uses all n available pseudorange measurements to obtain the GPS solution. The regressor, or **H** matrix, is the conventional matrix of direction cosines with ones populating the last column. The $n \times 4$ **H** matrix is a tall matrix so the generalized inverse is used resulting in a least squares solution.

4.2 Results

The results presented in this thesis are the cumulative representation of 5000 Monte Carlo (MC) runs. It was found experimentally that 5000 MC runs are enough for the average miss distance and its standard deviation to converge for the algorithms presented in this thesis. In order to provide an unbiased comparison between the results from each approach, the Gaussian pseudorange noise realization for each satellite is maintained the same between both algorithms for any given MC run. The estimation results as a function of satellite availability are shown in Table I. *Miss dist* is the experimentally determined three dimensional range between the true user position and the estimated position. The value shown in Table I is the average range over the 5000 MC runs. *std(Miss)* is the experimentally determined standard deviation of *Miss dist* over the 5000 MC runs. The predicted standard deviation of the miss distance is gauged according to:

$$\text{Predicted std(Miss)} = \sqrt{P^{+}_{\hat{u}_{11}} + P^{+}_{\hat{u}_{22}} + P^{+}_{\hat{u}_{33}}}$$

All *Miss dist* results have been normalized with respect to the measurement noise standard deviation σ. $\hat{\sigma}$ is the average of the predicted values of σ and std($\hat{\sigma}$) is the standard deviation of this average. Both $\hat{\sigma}$ and std($\hat{\sigma}$) have also been normalize with respect to σ. The number of iterations (*# iterations*) and *FLOPS* are the experimentally recorded number of iterations and FLOPS, required to produce the solutions, averaged over the 5000 MC runs.

	n = 6		n = 7		n = 8		n = 9	
	ILS Algorithm	2 Step Algorithm	ILS Algorithm	2 Step Algorithm	ILS Algorithm	2 Step Algorithm	ILS Algorithm	2 Step Algorithm
$\hat{\sigma} / \sigma$		0.81		0.89		0.93		0.940
$std(\hat{\sigma}) / \sigma$		0.60		0.47		0.40		0.35
Experimental Miss dist / σ	2.17	2.25	1.53	1.54	1.45	1.47	1.44	1.43
Experimental std(Miss) / σ	1.19	2.39	0.76	0.77	0.75	0.77	0.72	0.72
Predicted std(Miss) / σ		2.15		1.66		1.64		1.72
# Iterations	5	2.45	5	2.53	5	2.3	5	2.35
FLOPs	4115	3080	4535	3675	5013	4194	5503	5017

Table 1. Average Results from 5000 Monte Carlo Runs

4.2.1 Iterative Least Squares Algorithm Benchmark

The experimental average miss distance and its standard deviation produced by the ILS algorithm are used as a baseline for comparison to the algorithm presented in this paper. The average nondimensional miss distance is a function of the number of satellites in view and it ranged from 1.44 to 2.17. The experimentally obtained nondimensional standard deviation of the miss distance is relatively small and it ranged from 0.72 to 1.19. This relatively small standard deviation shows that the position estimates from the ILS algorithm are biased, and worse, relying on an experimental determination of the ILS algorithm miss distance standard deviation misleads one into trusting the ILS provided position estimate. Since the ILS algorithm does not provide a prediction of the estimation error covariance, this bias can cause serious problems during a straightforward integration of GPS position estimates from the ILS algorithm with Inertial Navigation System (INS) or SAR sensors data.

Irrespective of satellite availability, it took the ILS algorithm 5 iterations to converge to the required threshold for accuracy. Floating Point Operations (FLOPS) ranged between 4115 to 5503. The variance in FLOPS is a function of the size of **H** which changes as a function of satellite availability; of course the "miss distance" decreases as the availability of satellites in view increases. The ILS algorithm's FLOPS count would have been even higher had we ignored the results in [14], [25] and instead followed the conventional practice of selecting the best four satellites with the lowest GDOP. Moreover, the estimation results would have been poorer.

4.2.2 Two-Step Algorithm Results

From a performance point of view the novel two step algorithm produced results comparable to the baseline ILS results. As shown in Table I, the average miss distances yielded by the novel algorithm when compared to those yielded by the conventional ILS algorithm were similar, the difference ranging from 0.01 to 0.08. The experimentally determined standard deviation was slightly larger than that of the ILS algorithm, the difference ranging from 0.00 to 1.20. Most importantly, the predicted standard deviation provided by the new algorithm proved to be a good indication of the accuracy of the novel algorithm positioning estimate. For the case where $n = 9$, the predicted standard deviation of the miss distance called for 63% of the position estimates to be in an ellipsoid, centered at the true user position, with "radius" of $1\sigma = 1.72$ units. Experimental data showed 71% of the position estimates within the 1σ ellipsoid. For $n = 6$, the predicted standard deviation of the miss distance called for 63% of the position estimates to be in an ellipsoid, centered at the true user position, with "radius" of $1\sigma = 2.15$ units. Experimental data showed 58% of the position estimates within the 1σ ellipsoid. These results confirm the validity of the novel algorithm's estimation error covariance prediction Correct estimation error covariance information, viz., $P^{+}_{\hat{u}}$, is critical for the correct downstream integration of GPS positioning information and INS or SAR sensors data.

The novel two step algorithm takes 2 to 3 iterations to produce a position estimate and a prediction of the estimation error covariance while the ILS algorithm takes 5 iterations to produce only the position estimate. As a result, the FLOPS count for the two step algorithm is consistently lower than the FLOPS count for the ILS algorithm. Concerning the n dependence of the FLOPS count: Obviously, the miss distance decreases as n increases. Very good results are obtained for $n \geq 7$. However, even though the FLOPS count of both algorithms are proportional to satellite availability (n), the FLOPS count of the two step algorithm increases at a faster rate. This is due to the estimation of σ in the novel algorithm, which requires operations on $(n-1) \times (n-1)$ matrices.

Concerning the number of iterations in the two step algorithm: The two step algorithm could be hard wired to only two iterations instead of three to lower the FLOPS count even further. Experimental results show that if the number of iterations is reduced to two, the average miss distance and the estimation error covariance remain practically unchanged for $n \geq 7$ and change slightly for $n = 6$.

4.2.3 Unconventional Geometries

Given the deficiencies observed for the *closed-form* algorithm in typical near earth navigation scenarios, it is interesting to exercise the algorithm in unconventional high GDOP scenarios where the conventional iterative algorithm tends to have difficulties. This type of scenarios can be expected in WAAS and test range applications. In this paper the experimental test environment consisted of a simulated ground planar array of 36 pseudolites evenly spaced in a circular pattern with a 10000 meters radius and one pseudolite at the center. This pattern was selected because it represents the best achievable ground array for the iterative algorithm if the user is directly above the center pseudolite [25]. This number of pseudolites was selected to achieve satellite availability levels that allow for evaluation of the algorithm's data driven σ estimation capability. The simulations had the user positioned 10000 meters directly above the center of the circular pattern and the center pseudolite is moved away from the center to vary the geometry. In this test environment, the conventional ILS algorithm produces fairly good estimates of the four GPS parameters with a pseudolite directly below the user; however, the estimates quickly degrade as the center pseudolite is moved away from directly below the user and it fails to produce a solution when the center pseudolite is offset by more than 400 meters.

The *closed-form* algorithm (step 1) produces excellent estimates of the pseudorange measurement noise strength σ. The σ produced by the algorithm ranged from 0.98 to 0.9991. Unlike the results obtained for the typical near earth GPS scenario, the estimation of the two dimensional u_x, and u_y user position coordinates are extremely good, with errors smaller than those obtained with the conventional iterative algorithm; however, the user altitude (u_z) estimation error is very large ranging from 2.9×10^3 to 1.3×10^4. Given the extremely low estimation errors in the u_x and u_y user position coordinate estimates, it appears that the geometry produced by pseudolite ground planar arrays is more favorable to the *closed-form* algorithm than any geometry that can be produced considering strictly the 24 satellite NAVSTAR GPS constellation. If it was necessary to estimate only the planar u_x and u_y user position coordinates from signals obtained strictly from pseudolites in a ground planar array, as would be the case in a test range, the *closed-form* algorithm would be the algorithm of choice.

Taking the estimates produced by step 1 and applying the *Kalman update* algorithm in step 2 improved the estimate of u_z. However, the error in u_z is still too large to render its estimate useful. u_x and u_y are also slightly affected, sometimes for the worse, other times for the better. Based on the results, the *Kalman update* (step 2) does not prove very useful in this ground planar array test environment, as it does not provide any significant improvement over the *closed-form* algorithm (step 1). Moreover, the risk of corrupting the u_x and u_y position estimates exists. Hence, both the ILS and the novel two step algorithm do not yield good altitude estimates when our planar arrays of pseudolites is used and the center pseudolite is offset by more than 400 meters from the center of the array.

4.2.4 Closed Form (Step 1) Algorithm Results

Step 1 is a prerequisite for step 2 of the novel GPS positioning algorithm. Additionally, step 1 provides the estimate of the pseudorange measurement noise intensity (σ). Hence, in this Appendix step 1 is further discussed.

The experimental results indicate that the *closed-form* algorithm presented in this paper is extremely sensitive to noise. The sensitivity to pseudorange noise is reflected in the extremely large average miss distances which ranged from 47 to 120. The standard deviation ranged from 35 to 90. This is to be expected due to the high condition number of the regressor matrix. Note however that the *closed-form* results are merely the initial guess used to initialize the *Kalman Update* algorithms developed in this paper, and are not the final answer. Concerning the use of step 1 results in step 2 of the novel algorithm: these results are perfectly valid in view of their rigorous derivation using linear mathematics only. There is no concern of Kalman filtering divergence.

The results show that the condition numbers for the (n - 1) x 4 regressor, **H**, yielded by the *closed-form* algorithm are all extremely large, ranging between 600 and 775. Consequently, it is unlikely that this algorithm can provide a GPS solution with small errors. The ill conditioning of the regressor is largely due to the last column which is made up of the difference between the pseudoranges. The first three columns are made up of differences in the three satellite position coordinates (u_x, u_y, u_z) respectively which tend to produce much larger differences than the pseudorange differences. The poor scaling due to the last column of the **H** matrix manifests itself as extremely large errors in the range-equivalent user clock bias where the observed errors ranged between 1.83 and 7.75. The large errors in the clock bias estimates do not affect the position error which is strictly a function of the error in the estimated position coordinates.

An additional feature of the *closed-form* algorithm is its ability to provide a data driven prediction of the covariance of the GPS solution estimate. The prediction $\hat{\sigma}$, the standard deviation of the pseudorange measurement noise, is not reliable when only 5 satellites are available. However, as satellite availability increases, the prediction $\hat{\sigma}$ improves accordingly. The experimental results show that with six satellites in view, the average $\hat{\sigma}$ is 0.81. However the standard deviation of this prediction averages 0.60. With seven satellites the average $\hat{\sigma}$ is 0.89. The average standard deviation is 0.47 showing a lot of improvement. With eight satellites the average $\hat{\sigma}$ is 0.93 showing further improvement and the average standard deviation decreases to 0.38. Comparison of $\hat{\sigma}$ and the true σ ($\sigma = 100$ m) indicates that at least 6 satellites must be available before a reliable prediction of σ can be obtained.

5. Conclusions and Recommendations

This chapter presents a brief summary of the performance related issues of the novel 2 step algorithm. Emphasis is placed on identifying the areas of strength and suggesting applications for which it is best suited. The chapter sums up with recommendations for future work.

5.1 Conclusions

The performance of the novel two step algorithm is comparable to the performance of the baseline ILS algorithm and, furthermore, it retains all the attractive features that motivated the development of the *closed-form* algorithm in the first place. Considering the *closed-form* algorithm as supplemented by the *Kalman update* algorithm as a single two step GPS position determination algorithm, a novel algorithm with the following attributes has been developed:

1. The performance under typical navigation scenarios, using only the NAVSTAR GPS satellite constellation, is equivalent to the performance achieved by the conventional ILS algorithm used as a baseline.

2. The algorithm is closed-form, hence it can be used under any geometrical conditions without the need for externally provided initialization and a degree of autonomy is thus achieved.

3. The algorithm is computationally efficient due to its "non-iterative" nature and its lower FLOPS count.

4. The algorithm has the capability to produce a data driven estimate of the measurement noise strength (σ) and, most importantly, predict its estimation error covariance.

5. The horizontal positioning performance of the novel two step algorithm under poor geometry conditions, e.g. when ground-based planar arrays of pseudolites are used, is better than that of the conventional ILS algorithm. Moreover, there are no restrictions on the user position and an initial user position guess is not required.

The step 1 preliminary solution provided by the *closed-form* algorithm presented in this paper is extremely sensitive to noise. At the same time, and since linear mathematics are used, a good position estimate is obtained. Used in conjunction with the *Kalman update* algorithm in step 2, a GPS solution estimate comparable to the conventional iterative least squares algorithm is obtained. The preliminary *closed-form* algorithm's ability to produce a prediction of the estimation error covariance is a valuable asset which is essential for the initialization of the *Kalman update* (step two).

The strength of the *closed-form* algorithm surfaced in pseudolite ground array scenarios. In these scenarios the pseudolite availability is such that an excellent estimate of the pseudorange measurement noise strength, σ, could be recovered from the measurement residuals, which can then be used to calculate the estimation error covariance. The performance of the *closed-form* algorithm in estimating the horizontal user position parameters showed considerable improvement over the iterative algorithm; furthermore, no user position restrictions were required as long as the user was within the confines of the outer radius of the circular pattern. This may prove beneficial to test range applications where the conventional iterative algorithm is at risk of failure and this imposes restrictions on the flight test trajectory and altitude.

In conclusion, the benefits of the novel noniterative algorithms are computational efficiency, data driven predictions of the pseudorange measurement noise strength and the estimation error covariance, no need for an initial position guess, and better performance under poor geometry.

5.2 Recommendations

This section presents areas that remain to be explored that can be taken on as follow on research.

5.2.1 Alternate Stochastic Closed-Form Algorithms

New approaches to deriving alternate stochastic closed-form solutions to the system of pseudorange equations must be investigated in an attempt to obtain an algorithm that possesses the following qualities:

- The regressor matrix should have a low condition number to maintain the estimation error amplification bounds to a minimum.
- The algorithm should be capable of producing an estimate of the four GPS estimation parameters using only four pseudorange measurements.
- The algorithm should be capable of producing an accurate GPS solution with a single application without the use of a supplementary algorithm.

It must be noted that the existence of, or feasibility of developing, an algorithm that possesses all or any of the above qualities is not guaranteed.

5.2.2 GPS Measurement Noise Levels Investigation

The accuracy of the novel algorithm is very sensitive to the accuracy of the measurement error estimate. New, more accurate, approaches for predicting the measurement noise strength must be investigated. Moreover, an new algorithm, capable of predicting the measurement noise strength when only 5 satellites are in view, must be developed.

Another area that remains to be explored is the comparison between the existing algorithms that predict the measurement noise strength using the measurement vector and regressor matrix of the ILS algorithm and the algorithm presented here.

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The use of detect, or closed-form solutions of the trilateration equations used to obtain the position fix in GPS receivers is investigated. The paper is concerned with the development of an efficient new position determination algorithm that uses the closed-form solution of the trilateration equations and works in the presence of pseudorange measurement noise and for an arbitrary number of satellites. In addition, an initial position guess is not required and good estimation performance is achieved even under high GDOP conditions. A two step GPS position determination algorithm which 1) entails the solution of a linear regression problem and, 2) an update of the solution based on one nonlinear measurement equation is developed. The closed-form solution of the linear regression in step 1 provides an estimate of the GPS solution, viz., user position and user clock bias, as well as the estimation error covariance. In the update step 2, only two to three iterations are required, as opposed to five iterations which are normally required in the standard iterative least square algorithm currently used in GPS. The two step algorithm also provides a data driven prediction of the pseudorange measurement noise strength and the estimation error covariance. The mathematical derivation of the novel and efficient solution algorithm for the GPS pseudorange equations using stochastic modeling is validated in a realistic simulation experiment based on 5000 Monte Carlo runs. The algorithm's performance is discussed and compared to the conventional iterative least squares algorithm currently used in GPS.

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