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Decision Factors for Cooperative Multiple Warhead UAV Target Classification and Attack with Control Applications

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DISSERTATION

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AFIT/DS/ENY/05-04

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At this point, I hope the reader won't mind me getting a little more personal when I say that I would also like to thank my Mom and Dad. Thanks for being such great parents and instilling in me the love for learning. You two are a great inspiration.

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Douglas Dwayne Decker

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Nomenclature

A	The region covered by UCAV by time τ or normalized time/space x
$\mathsf{A_m}$	Region left after subtracting A from $\mathsf{A_s}$
$\mathsf{A_r}$	Circular battle space region
$\mathsf{A_s}$	Rectangular battle space region
$\mathsf{A_y}$	Sub-region of $\mathsf{A_m}$
$\mathsf{A_z}$	Sub-region of A
A	Area for region A
A_m	Area of the region $\mathsf{A_m}$
A_r	Area of the region $\mathsf{A_r}$
A_s	Area of the region $\mathsf{A_s}$
A_y	Area of sub-region $\mathsf{A_y}$
A_z	Area of sub-region $\mathsf{A_z}$
α	Density of FT's in $\mathsf{A_s}$
α_A	Density of FTA situations in $\mathsf{A_s}$
β	Density of T's in $\mathsf{A_s}$
β_A	Density of TA's in $\mathsf{A_s}$
f	Number of FTA's
$f_{t,f}^{(w)}(x)$	Associated Probability Density Function of $p_{t,f}^{(w)}(x)$
f_y	Number of FTA's in $\mathsf{A_y}$
$\mathcal{F}_{f,X}$	f FTA's in X
$\mathcal{K}_{t_k,X}$	t_k target kills in X
$\lambda_{A_{FT}}$	Poisson parameter representing average number of FTA situations that would occur in $\mathsf{A_s}$
λ_{A_T}	Poisson parameter representing average number of TA's in $\mathsf{A_s}$

λ_{FT} Poisson parameter representing average number of false targets in A_s

λ_T Poisson parameter representing average number of T's in A_s

M Number of false targets in A_s or A_r

m Number of TA's the user defines as mission success

N Number of targets in A_s or A_r

$NRFT_f$ The event that exactly f of the FT's are not recognized (i.e. they are mistaken for T's)

NRT_t The event that t targets are not recognized for what they are

P_{FTR} Probability of false target report given a false target was encountered

P_k Probability of a target kill given a target attack

$P^{w_A}(x)$ Probability of w_A warheads remain after A

$P_{t,f}^{(w)}(x)$ Probability of t TA's and f FTA's in A assuming we have w warheads. If t is missing, it indicates we are only concerned with the number of FTA's (regardless of the number of TA's). If f is missing, we only care about the number of TA's.

$P\left(X_{t,f,x}\right)$ Another notation for the probability of t TA's and f FTA's in A assuming we have w warheads. If t is missing, it indicates we are only concerned with the number of FTA's (regardless of the number of TA's.

P_{TR} Probability of target report given a target was encountered

P_{TR_B} The P_{TR} for the ROE that both UCAV's must classify an object as a target before its declared a target

P_{TR_E} The P_{TR} for the ROE that either UCAV or both UCAV's can classify an object as a target for it to be declared a target

$P_{\mathcal{F}_f}(x, \Delta x)$: Instantaneous transition probability of a FTA in Δx given there were exactly f FTA's by x. Used in the Markov derivations.

$P_{\mathcal{T}_t}(x, \Delta x)$ Instantaneous transition probability for a TA in Δx given there were exactly t TA's by x. Used in the Markov derivations.

q The parameter defining a ROC curve

r Ratio of λ_{FT} to λ_T

RFT_f The event that f false targets were recognized for what they are

RT_t The event that t targets were recognized for what they are

σ^2_{FT} Variance of the location of the false targets in $\mathsf{A_r}$

σ^2_T Variance of the location of the targets $\mathsf{A_s}$

t_k Number of target kills

t Number of TA's

t_y Number of TA's in $\mathsf{A_y}$

$T_{j,X}$ j Targets in X

$\mathcal{T}_{t,X}$ t TA's in X

τ Time into the mission

Υ Time to search region $\mathsf{A_s}$ or $\mathsf{A_r}$

V Velocity of UCAV

w_A Number of warheads remaining after the sweep thru A

w Number of warheads on a single UCAV

$\mathcal{W}_{\mathcal{A}}$ w_A warheads left after A

W Width of UCAV search footprint

x Normalized time or space

x_c The time a UCAV is away from its assigned area (the time to cooperate)

x_d Mission duration

x_f Time of mission failure

x_s Time of mission success

y Normalized time associated with $\mathsf{A_y}$

z Normalized time associated with $\mathsf{A_z}$

List of Abbreviations

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Abstract

Autonomous wide area search, classification and attack using Unmanned Combat Air Vehicles (UCAVs) is considered. The wide area search and attack scenario is modelled, capturing the important problem parameters of target density in the battle space, the density of false targets, the seeker and Autonomous Target Recognition (ATR) modules' performance parameters, as well as munition parameters such as search rate, time, and warhead lethality. The analysis in this research is an important stepping stone towards establishing benefits of cooperative search and engagement in a multi-vehicle scenario. This research uses probabilistic analysis to formulate and analytically solve for the probability of success in search and engagement as well as probabilities of other events of interest. Two methods are used to compute these probabilities. The first method utilizes a detailed examination of the sub-events required for the event of interest to occur. The second method utilizes a Markov chain approach. In each method, general expressions are first obtained that are applicable to any assumed *a priori* distributions of targets and false targets. These expressions are subsequently applied to a multiple warhead munition/UCAV operating in a single target/multiple false target scenario and then several multiple target/multiple false target scenarios. This research shows how the analytically derived results can be applied to all facets of the balanced system design and operation of Wide Area Search Munitions (WASM) including the evaluation of cooperation schemes and rules of engagement. This dissertation also formulates the problem as a control problem and examines the possibility of utilizing this formulation in the real-time estimation of the target and false target distribution parameters.

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I. Introduction and Literature Search

1.1 Overview

Air-to-ground warfare has seen an evolution from unguided gravity bombs to modern day "smart" bombs guided by lasers and/or Inertial Navigation Systems (INS) aided by the satellite based Global Positioning System (GPS). Imaging terminal seekers, originally lock-on before launch, are now capable of autonomous target acquisition and can now be combined with way-point INS/GPS mid-course guidance to provide autonomous wide area search air vehicles or munitions. Possible future concepts include cooperative Wide Area Search Munitions (WASM's) acting in hunter-killer packs to find dispersed targets and converge on identified targets to deliver sufficient lethality to accomplish mission objectives. While the potential utility of these concepts is often acknowledged, there is insufficient analysis to support an exhaustive evaluation of the effectiveness of these concepts. Specifically, decision factors such as probability of success (kill), expected number of kills and expected number of false target attacks are needed for evaluation of alternative concepts and, eventually, operation resource allocation. Much of the work done so far has concentrated on simulation studies that quantify results for specific scenarios, but often do not provide the broader underpinning for a thorough understanding of the design and operational employment aspects of the problem. More analytic work is needed to define the fundamental nature of the wide area search munition problem, to include identification of the critical munition and target environment parameters that must be adequately modelled for a valid simulation.

Along with the concept of autonomous decision making is the ability to have the UAVs work together cooperatively. Cooperation among multiple UAVs could result in a more comprehensive and thorough search, more accurate classification, and more effective attack of targets. If done in an autonomous fashion (the vehicles conduct those tasks without human intervention) we would have, in effect, a fire and forget capacity for a fleet of UAVs which could search an area and destroy any targets found (or go after a prioritized list of targets). In fact, "advanced autonomous fault-tolerant guidance and control algorithms for multiple UAVs in conjunction with effective decentralized multi-agent coordination strategies are of great interest to DoD [Department of Defense]" [41].

1.2 Previous Work

Much work has been done in the areas of search, classification, and attack of a target using cooperative control. For our purposes, we will classify the work done to date according to several categories. The first category which naturally comes to mind is work done in search theory. In fact, this is the field most closely related to this author's work. It is also the field where much analytical work has been done. However, there are still some holes when we look at the problem from an attack perspective. Specifically, the works in search theory, including classic works by Koopman [36], Stone [55], and Washburn [62], do not specifically address scenarios with multiple targets/false targets and a multi-warhead vehicle. Richardson [48], Stone [54], Benkoski [8], and Stone and Washburn [58] wrote surveys covering the search literature. Problems discussed included stationary target problems, moving target problems (to include evading targets as well as non-evading targets), optimal search density problems, and optimal searcher paths.

Richardson [48] classified the work done in search theory up to that point according to assumptions made in measures of effectiveness (probability of detection, expected time to detection, probability of correctly estimating target location), target motion (stationary, Markovian, diffusion), and characterization of search effort (continuous or discrete). While this summary concentrated on detection of a target, these classifications work for attack as well. In addition, when discussing the attack of target(s), we can introduce further classifications based on assumptions made in the number of searchers (in our case, UCAVs), whether each UCAV carries a single warhead or multiple warheads, and the number and distributions of targets and false targets present.

It should be noted that much of the work done to date has concentrated on optimizing some search criterion. Typically this involves varying the "search effort" (the independent variable, typically constrained to be no more than some quantity) to maximize some criterion which usually depends on a detection probability which is a function of the search effort. A common criterion might be the probability of detection itself. Typically, the detection probability is such that an object will not be detected with zero search effort and is certain to be detected with infinite search effort.

1.2.1 Discrete Search, Single Searcher. Typically, a discrete search is such that the stationary target(s) are each located in one box/cell out of n boxes/cells. Several authors concentrated on searching for a single stationary target. Tognettie [60] concentrated on knowing the "whereabouts" of his target, but was not concerned with physically locating the target. That is, the objective is to maximize the probability of correctly stating in which area the target is located. One can either find the target or, after an unsuccessful search, correctly guess in which area the target is located. He found that when limited to n searches, the optimal strategy was to partition the searches such that one area is missed. His work incorporated multiple "looks" at each area (a look is a single search in an area with a conditional probability of detection that is constant for that area). Arkin [3] allowed for simultaneous looks at multiple areas, but the more areas that are searched simultaneously, the less the probability of detection.

Kadane [30] extended Tognettie's work by incorporating a cost for search. He examined searches where, if no target is found during the search, then box (area) i will be declared to contain the target. He determined that for that type of search, the optimal search will not include box i. In addition, he developed an algorithm for finding an optimal whereabouts-search strategy. Trummel and Weisinger [61] incorporated an *a priori* probability distribution for the target location. They showed that finding an optimal searcher path that maximizes probability of detecting the target by the end of a fixed time is NP-complete, while minimizing the mean time to detection is NP-hard. Hall [21], [22] also incorporated an *a priori* target location distribution and introduced the concept of using a random variable for the probability that a particular search of a particular area will miss a target that is really there.

Similar work has been done for multiple targets by Smith and Kimeldorf [52], Kimeldorf and Smith [35], Assaf and Zamir [4], and Kelly [34].

1.2.2 Continuous Search, Single Searcher. Most of the more modern work has concentrated on the continuous search problem. Currently, emphasis is being placed on continuous search of a moving target.

Most of the work addressing search for stationary targets date back several decades. Supposedly the stationary target problem has "reached a mature state" [58] with little expectation of significant extensions. However, we contend that this is not the case once we incorporate a limited number (greater than 1) of warheads on a UCAV seeking a target or targets. Most, if not all, of the work done in search theory in effect assumes either a single warhead (search is complete upon finding the target) or a limitless supply of warheads (search continues until all targets found and distinguished from any false targets encountered).

To try and make sense of the literature in this area, and to point out the area of our contribution, we have delineated this category even more by the number of targets (single or multiple) and the number of false targets (none or multiple). We can then look at other categories, such as optimization, cooperation and some control type categories.

1.2.2.1 Single Target, No False Targets. Hoai and Leondes [23] sought to maximize the detection probability of the target using a single-try (non-redundant) search which is a function of the search effort and the location of the target. Their search effort is a function only of the location of the searcher and so uses Dobbie's [14] extension of Koopman's exponential detection law. The main point of their work was to try and eliminate the need for knowing the target's *a priori* probability density function (pdf). They concluded that the "minimax solution guarantees a positive detection probability at the expense of degradation in performance". Performance here is defined as the probability of detection.

Iida et al. [25] noted that studies on the optimal distribution of search effort had consistently made an assumption of "local effectiveness of searching effort". In this assumption, the searching effort at a point is assumed to only be able to detect an object at that point and not an object which is in the neighborhood of the point being searched. Their work eliminates that assumption. They also used an exponential detection function similar to Koopman's random search formula and assumed that the search effort was only dependent on location and not time. In other words, once the searcher looks at an area, the decision is instantly made on whether an object was detected or not.

Many years earlier, Richardson and Belkin [49] looked at the sensor's effectiveness as well, but their work concentrated on the effect of an uncertain, fixed sweep width. De Guenin [12] provided a method of solving the problem of allocating a given amount of search-effort to maximize the probability of discovering the object without any assumption on the form of the detection probability function. This function is the probability of detecting the target at x given the target is at x using some search effort at x ($\phi(x)$). This detection probability is then $p[\phi(x)]$. The object's location is a random variable, X, with p.d.f. $g(x)$:

$$g(x)dx \quad = \quad P\{x \leq X \leq x + dx\}$$

The objective is to find $\phi(x)$ which maximizes the probability of detecting the target

$$P = \int_{-\infty}^{\infty} g(x)p[\phi(x)]dx$$

with the total search effort, $\phi(x) > 0$, constrained to some amount

$$\int_{-\infty}^{\infty} \phi(x) = \Phi \ .$$

De Guenin's method for finding $\phi(x)$ does not depend on a particular form for $p[\phi(x)]$, uses successive approximations, and gives a unique $\phi(x)$. The search is not necessarily exhaustive, however, but concentrates on most likely locations of the target.

1.2.2.2 Single Target, False Targets. Once we allow for false targets, a detection function is not enough; we must also be able to classify the object that has been detected as either a target or false target. A modelling decision must be made. Do we assume all objects will be detected (eliminating the detection function - leaving only the classification function) or do we keep both the detection function and classification function?

Stone and Stanshine [57], [56] were among the first to look at this issue. They chose to model the process of finding a target using a detection process (scan the area with the sole intent of detecting an object with no distinction between the intended target or a false target) and then a classification process which determined if the contact was a false target or the intended target. In their model, the classification was certain but only after a finite, although random, amount of time. In their earlier work, [57], this process could not be interrupted once begun. Later, [56], they relaxed that requirement. The detection process, however, was not certain. The probability of detection of an object was, in fact, a function of the search effort applied (they call it the "broad search density function") and is what we are calling in this work the detection probability function (they called it the "local effectiveness function"). The search effort was a function of time and location of the target. They then examine the problem of minimizing the mean time to find (contact and classify) the target. Note that the probability of classifying a false target as a target is zero. They note the similarities in their concept of broad search detection process to De Guenin [12] and their concept of their search effort function to Arkin [2].

In their follow-on paper [56], Stone and Stanshine lift the restriction that an investigation cannot be stopped once initiated. In fact, they allow for a designated maximum amount of effort applied to the investigation process. Either the object is correctly identified after that amount of effort or the searcher moves on. Again, no misclassification of a false target is allowed. They also apply Richardson's problem of an uncertain sweep width to their problem.

Dobie [15] noted that Stone and Stanshire's work assumed that their search plan did not depend on the number of false targets found. His work looked at allowing search plans which do depend on the number of false targets found. However, he limits his study to problems where the number of false targets are bounded and was able to obtain the solution only for a particular case. In their search plan, the same spot can be searched more than once (duplicative search). The Stone papers could also be interpreted this way; however, using the interpretation that it is a duplicative search, they do not make use of the number of false targets found in their search plan. They merely mark the FT so it will not be investigated again. Dobbie's plan also marks the FT to ensure no duplicative investigation, but he also utilizes any information concerning the number of false targets found to that point. "The optimal plan in our class of search plans depends on the number of found false targets" (Dobbie, pg 913). In fact, each time they detect a false target they create a new search density (the search effort applied to each spatial increment in the entire area). Again, as with Stone, they assume that given an infinite amount of search effort, they will find the target.

Klabaugh [31] looks at scenarios similar to Stone and Dobbie, but models the classification as an instantaneous process which is reliable only with a given probability. His search plan could not be modified when false targets are located and correctly classified.

Iida [24] looks at a two-stage search (broad and investigating). His false contacts are only caused by system noise - meaning the investigating search gives no further information. This means the investigating search must be abandoned at some point. He then tries to find the optimum time for the investigating search of the contact. He assumes the signal is such that it can be determined if it is from a true contact or noise, but only with a given probability. He also restricts his total search time to some number.

1.2.2.3 Multiple Targets, No False Targets. Cozzolino [11] looked at continuous search with multiple targets of differing sizes and no false targets. He also looked at the problem from a probability of detection and classification (with regards to size) as a function of search effort. He in effect assumes a limitless supply of warheads since he assumes a Poisson distribution of targets and he has the capacity to find any number of them. His results include the probability distribution of the number and sizes of discovered objects, and the prior and posterior distributions of the number of objects remaining undiscovered. The states of his system are the number of objects in an area and their sizes. It could be noted here that although he gives equations for the probability of contacting a target by a given time, that this cannot be directly related to probability of attacking a target by a given time. Since they are only examining the search for targets, they have no limitations on the number of false targets misclassified. It simply gets marked as a target (incorrectly) and they continue on. However, the probability of attacking a target is dependent on the number of misclassifications of false targets (all the warheads could be used before getting to the real target).

1.2.2.4 Multiple Targets, False Targets. Jacques and Pachter [26], [29], [27] have derived an analytic solution for search and attack probabilities when multiple targets and false targets (all stationary) are present. Their work concentrated on a single munition searching a region $\mathsf{A_s}$ of area A_s. In their work [29], analytic solutions for six scenarios of interest were derived. The scenarios are described as follows (in all scenarios the targets and false targets are stationary):

Scenario 1: A single target uniformly distributed throughout $\mathsf{A_s}$ and a Poisson field of false targets.

Scenario 2: Poisson field of targets and a Poisson field of false targets.

Scenario 3: N targets uniformly distributed, and a Poisson field of false targets.

Scenario 4: N uniformly distributed targets, and M uniformly distributed false targets.

Scenario 5: The battle space consists of a circular disc of radius r centered at the origin. There are N targets, distributed according to a circular normal distribution centered at the origin and a Poisson field of false targets.

Scenario 6: Same battle space as Scenario 5, with N targets distributed according to a circular normal distribution and M false targets distributed according to a circular normal distribution.

This research extends the work of [29] by incorporating multiple warheads on a single UCAV searching a field of multiple targets and/or false targets. Using a classification system similar to Richardson's [48], this dissertation examines the probability of attack and probability of kill during a continuous search for stationary targets among multiple false targets using a single multi-warhead UCAV.

1.2.3 Cooperation.

1.2.3.1 Cooperative Search. Much work has already been done dealing with cooperative search. Polycarpou et al. [47:Ch 13] focus on cooperative search in which they seek to follow a trajectory that would result in maximum gain in information about the environment (but it could easily be extended to cooperative engagement and classification, etc). The only cooperation between agents is the sending of the information they have. No agent tells another what to do nor are there any negotiations between agents. Each seeks to enhance a global goal (not only its own goal). They call it passive cooperation. It has the advantage that it is robust to loss of any particular vehicle. Simulation seemed to be the evaluation tool of choice.

Yang et al. [64], examined cooperative search using an opportunistic cooperative learning method. This method is used to update a Target Probability Map (TPM) using sensor readings taken in each cell during the search. A Bayesian update rule was developed to determine the posteriori probabilities. The TPM is initialized with *a priori* knowledge about possible target locations and is updated as the UAVs take their sensor readings. All UAVs have access to the TPM. The goal of the cooperation is to reduce the uncertainty of the target locations as rapidly as possible. Their reward scheme has the tradeoff between trying to explore the environment (in which they try to cover the whole environment as rapidly as possible) and covering target rich areas they believe have the highest probability of finding targets. Simulation was used for evaluation.

Flint et al. [18] formulated the problem in terms of multiple UAV's that must generate their own paths to maximize the number of targets which are positively identified. They formulated a discrete time stochastic decision model which they then implemented using a dynamic programming algorithm.

Bethel and Paras [9] have looked at a "front-end" detector configuration in which an area is scanned. Targets in that area are said to be in one of M bins which make up that area. The idea is that of a radar in which bearings to the target(s) are recorded. The bins are defined by the bearing boundaries. They determine posteriori probabilities that potential target l ($l = 1, 2, \ldots, L$, the max targets the systems can track) is present in the scan (regardless of bin location) and the posteriori probability that that target is in a certain bin given that the target is present in the scan. They can then multiply the probabilities and compute the probability that target l is in bin m_l. However, these probabilities cannot be computed directly and must be approximated using a a multi-target tracking system with individual detector loops and individual tracker loops which uses a discrete pdf linear Kalman-Bucy filter. Theoretically, the desired probabilities could be determined from more basic *a priori* probabilities, but the required computations grow exponentially with the number of targets they wish to track. For that reason, they chose the approximation method mentioned previously. They later extend their work to multiple sensors, but approximation is still required. Our work does not assume a given *a priori* distribution of targets or false targets.

Genetic algorithms have been used to develop decision rules for UAVs to "maximize the information gained by the UAV during its period of operation" [42]. These rules are developed by running many simulations and modifying the rules based on those simulations; however, all these searches are based on posteriori probabilities of targets given the observations they have encountered and depend heavily on simulation to analyze their method. A proper analytical probabilistic framework could, among other things, help verify these simulations.

1.2.3.2 Cooperative Classification And Cooperative Attack. Pachter and Hebert [45] tackled the cooperative classification issue by assuming a rectangular target with a known and measurable ratio of side lengths. With a given rectangle, their work shows optimal look angles for classification. In addition, for two UAVs cooperating, it shows the optimal angular separation for the second look. They then find the minimum time trajectory to achieve the optimal look angle, given a specific starting point. They do this for the UAV and for the case where the UAV has a sensor with a circular footprint of radius r whose center is d units in front of the UAV. In the latter case, the end point is the target location and the objective is to get the target within $r + d$ units of the UAV with the UAV looking right at the target. This work would be useful once a UAV has asked for confirmation from another UAV. The UAVs could collectively determine which one is closest and which one is least likely to find another target and do tradeoff calculations to determine which UAV will conduct the second look (or attack).

Chandler and Pachter [10] looked at cooperative classification and cooperative attack. Multiple views were combined statistically until sufficient confidence was reached. Nearby vehicles calculated trajectories and costs to all the objects and were assigned optimally.

When two or more vehicles are utilized to search and attack, a decision must be made as to whether we continue the search or go attack previously found targets. This leads us to examine work done in an area called optimal stopping.

1.2.3.3 Cooperative Search, Classification, and Attack. Nygard, et al. [44] have proposed a method for dealing with a “swarm of air vehicles whose mission is to search for, classify, attack, and perform battle damage assessment”. In their scenario, each UAV has a single warhead and can communicate and receive target field information to and from all the elements of the swarm as it becomes available. The result is an integer programming problem formulation that results in solutions that are globally optimal and can be computed locally and independently. To do this, though, one must accurately specify cost functions.

While some work has been done on aspects of cooperation (particularly in [28]), more needs to be done. Jacques [28] initially limited his analysis to Scenario 1 (single target uniformly distributed, Poisson field of false targets) with multiple UAVs. He considered two different path formulations. One where two UAVs follow the same path, and the other where two UAVs followed opposing paths. He stated that a general formula for the probability of mission success (killing the target) for N munitions (i.e. UAVs) has yet to be defined. In each case, simplifying assumptions were made that the UAVs were identical and that their behavior when searching over the same path was uncorrelated. Some of his students then continued his work. In particular, Park [46], Dunkel [16], and Gozaydin [20] examined Scenario 2. In addition, Jacques [28] made some forays into the cooperative classification and attack arenas, but again, concentrating on Scenario 1. Pachter [45] also looked at the classification problem. His work was described in the previous section.

1.2.4 Optimal Stopping. When dealing with the topic of search, the inevitable question is when to stop that search. In the literature this has become known as Optimal Stopping. We will not deal with this topic in this work. Some works related to optimal stopping are due to Willman [63], Starr [53], Bather [7], and Glazebrook [19]. Keeney [33] wrote an informative article dealing with the subject of trade-offs in general.

1.2.5 Control Formulation. Finally we will look at areas related to putting this topic into a control formulation. We have a system in which the states of the system could be defined as the number of target attacks and the number of false target attacks. The UCAV's could be the system's sensors and actuators. The objective could be to reach a certain state or maximize the number of target attacks or kills.

1.2.5.1 Markov Model. Work has been done exploiting the Markovian nature of scenarios similar to the ones we will propose; however, the analysis has tended to concentrate on duels in which there is either a pursuer/evader relationship or a battle between Red and Blue forces.

Kress [38] claims to the be the first to have derived state probabilities for the many-on-one duel. His work treated the time to kill as the random variable. He looked at a negative exponential distribution on the many side and a gamma-distribution on the one side. His model had N Red units on a single Blue unit B in which the N Red units fire continuously and independently of each other.

Feigin, et al. [17] proposed a continuous time homogeneous Markov model (transition probabilities do not change over time) for analyzing M on N air combat. Their states consisted of the number of free blue and free red planes and the number of pursuing blue and pursuing red planes (4 states). They base their Markov model on the following parameters: detection/advantage acquiring parameter, the average rate at which a pursuer reaches firing position and fires, the kill probability of a single weapon release, and average evader's disengagement rate. They used this model to evaluate acquisition type decisions and determination of optimal force size for multiple engagements.

Koopman [37] looked at the problem mainly from a cost (logistical) perspective. However, he did examine Markovian systems involving duels (opposing forces detecting and attacking each other). His work provides a good description of how to determine transition rates.

Barfoot, [6] looked at Markov duels in which the outcomes of shots by each weapon form a Markov process. Their work concentrates on the outcome of the final end game (the firing of rounds). They fire volleys in rounds (each fires a volley, then a given time later they fire again). The interval between firings is constant. Here he extended work by Ancker and Williams [1] who assumed the outcome of a shot was either 'killed' or 'not killed'. In Barfoot's work, the outcome of the round consists of a combination of events, whether the round hit, whether it killed, and whether the shooter senses the round missed (and where the missed round went).

Work was done in a non-duel sense by Sung and Sohn [59] who examined a system of multiple stand-by Remotely Piloted Vehicles (RPVs) and a single battery against a single passive enemy target. It was the first to consider such a combined system which works against a target kill. (The other works prior to this paper looked at direct duels.) They determined several combat measures of effectiveness to include time-varying mean and variance of number of RPVs being alive and of surviving enemy target attack, mission success, mission failure, mean and variance of combat duration time. It used the RPVs serially. They were in stand-by until the single RPV tracking the target was killed. Then a single RPV went out of standby to replace the destroyed RPV. The states are denoted by number of remaining RPVs, target alive or dead, RPV has sent target location to battery or not.

1.2.5.2 Posteriori Observations. Mahler and Prasanth [41] are proposing an ambitious research program which will 1) develop a mathematical programming framework for hybrid systems analysis and synthesis, 2) develop a computational hybrid control paradigm, 3) develop transition-aware anytime algorithms for time-bounded synthesis, 4) develop suitable modelling and cooperative control of UAV swarms for a SEAD-type mission, 5) develop new theoretical approaches for integrating multiplatform, multisensor, multitarget sensor management into hybrid systems theory, 6) investigate real-time nonlinear filtering for detecting and tracking low-observable targets, 7) develop new approaches to distributed, robust data fusion. They split their work into two categories; Multi-Agent Collection and Mutli-Agent Coordination. In Coordination, they claim existing approaches can be divided into three categories: the leader following, behavioral, and virtual structure approaches. They will look at ways to control a UAV formation with a novel integration of all three approaches. In Collection, they claim that until recently there has been no systematic, rigorous, and yet practical engineering statistics upon which to base multisensor, multi-object tracking. As a result, they believe progress has been hampered in "multisensor-multitarget data fusion, detection, tracking, and target identification. This lack has also probably hampered the development of systematic, control-theoretic approaches to sensor management, distributed sensor management, and multiplatform coordination" [41]. They propose using Finite-set statistics (FISST) to be that basis. They expect their research will address many, if not all, of those gaps.

Mahler's Multi-Agent Sensor Management seems to be the most closely related to the control formulation aspect of our research. They define sensor management as the process of "redirecting the right data-collection source at the right place or platform to the right target at the right time." They also say sensor management is inherently a stochastic multi-object problem (groups of targets, groups of sensors, groups of platforms, whose states and numbers can and do vary randomly in space and time). Their approach is to treat the Multi-Agent Collection and coordination process as "what it actually is ... a problem in nonlinear adaptive control theory in which both the data sources being controlled and the targets being tracked by the control process are, mathematically speaking, multi-object systems" [41]. FISST is an intriguing concept which could be an alternative to the control formulation we will propose.

A subset of the FISST concept is the Joint Multitarget Probability (JMP). Kastella presents "an approach to detection, tracking, classification and sensor management based on recursive evaluation of a joint multitarget probability" [32]. This probability is the conditional probability that there are exactly n targets of class c, located in cells x, based on a set of observations Z. His work looks at a one dimensional field using one sensor which can either detect a target or classify a target (but not both at the same time). The sensors update the *a priori* distribution (uniform distribution). For his model problem, he had two target classes with an unknown number of targets. The targets move independently with Markov transitions to nearest-neighbor cells. The JMP tends to be calculation intensive. Musick presents "a possible approach to the implementation of Joint Multitarget Probability based on a product approximation for the JMP equations" [43].

1.2.5.3 Parameter Uncertainty. In each of the six scenarios defined in Section 1.2.2.4, the distributions are characterized by a few parameters. These parameters are assumed to be known but, in fact, are not. Krokhmal et al. [39] addressed uncertainty in various parameters by using a Conditional Value at Risk (CVaR) methodology. They looked at a Weapon-Target Assignment (WTA) problem and used CVaR to minimize a loss function while ensuring a specified minimum probability of kill. Their uncertain parameters were the probability of kill for given weapons, and the number of targets in the battle space. Their control was the number of weapons each vehicle used to attack a target.

1.3 Research Statement

Previous work has concentrated on simulations with some work towards analytic expressions for some key probabilities. However, these studies have been limited either by the number of targets, false targets, and/or warheads. This research will focus on developing the analytic equations for various probabilities and expected values for UCAV's with multiple (finite) warheads for six scenarios. We limit our search to a continuous, exhaustive, and non-duplicative search for multiple stationary targets amongst a field of multiple stationary false targets. Targets and false targets have distinct *a priori* distributions. In terms of previous work in the area, we assume constant search effort throughout the area with the probability of detection of the targets (and false targets) equal to one and a given probability of correct classification. We assume a single type of target. The probabilities and expected values for which we will provide analytical expressions follow:

- Probability of an exact number of target attacks and false target attacks

- Probability of a specified number of target attacks (and false target attacks) and their expected values
- Probability of certain number of warheads remaining after a region is searched
- Probability of additional target attacks given a certain number of warheads remaining after a region is searched
- Probability of mission success and expected time of mission success
- Probability of mission failure
- Expected vehicle longevity

We will compute these probabilities for the following scenarios:

- Scenario 1: A single target uniformly distributed throughout $\mathsf{A_s}$ and a Poisson field of false targets.
- Scenario 2: Poisson field of targets and a Poisson field of false targets.
- Scenario 3: N targets uniformly distributed, and a Poisson field of false targets.
- Scenario 4: N uniformly distributed targets, and M uniformly distributed false targets.
- Scenario 5: N targets distributed according to a circular normal distribution centered at the origin amongst a Poisson field of false targets.
- Scenario 6: N targets distributed according to a circular normal distribution amongst M false targets also distributed according to a circular normal distribution.

With these analytical expressions, we can then show various applications to include a method to evaluate cooperation schemes and rules of engagement. We also will put the problem of search, classification and attack of targets which are distributed amongst multiple false targets into a control formulation. With this formulation we can examine a possible method to conduct real time estimation of the parameters defining the distribution, specifically for Scenario 2.

1.4 Applicability

Results of the type discussed in the previous sections may be used to validate simulation models and to guide the development of tactical algorithms for cooperative search and engagement. In addition, the analytic framework may be utilized to make acquisition, design, operational, and tactical decisions. Acquisition decisions may come in the form of determining cost effectiveness and trade studies such as deciding whether to spend money improving sensors, warheads, or acquiring more UCAVs. Design decisions include the establishment of an operating point to balance probability of detection with a desire to keep false target attacks to an acceptably low level. Operational decisions may include deciding the number of UCAVs (or the number of warheads on a single UCAV) to send to a battle space given a probable number and/or location of targets or false targets. Tactical decisions could conceivably take place within the UCAVs themselves. For example, given the elapsed time of the mission, the UCAV could determine the benefit of continuing the search (i.e. determining the likelihood of finding another target in the time remaining) versus the benefit of assisting in classifying and/or attacking a previously discovered target (perhaps from another UCAV). It would then decide which alternative is more profitable and take the appropriate action.

1.5 Outline of Document

Chapter II describes the two types of battle spaces that are considered; rectangular and circular. It then defines the six scenarios considered in the probabilistic analysis.

Chapter III describes, in generic terms, the first of two methods used to develop the probabilities. This first method finds the probability of the "last attack of interest" occurring at x. The last attack of interest is the last attack which could define the event in question. We then integrate that probability over the battle space to find the probability of that particular event. This method is called the sequential event method. This generic description of the probabilities apply no matter the distributions assumed for the targets and false targets. We will find that most of the probabilities in which we are interested can be easily calculated using the probability of exactly t target attacks and f false target attacks by x.

In Chapter IV, we use those generic descriptions of the probabilities and apply them to the six scenarios. We calculate seven probabilities (or expected values) for Scenarios 1 and 2 and then concentrate on the critical probability, $P_{t,f}(x)$ for the other scenarios.

Chapter V describes the second method (using Markov chains) to find these same probabilities and then calculates them.

Once we have the probabilities, we show some examples of applications for them. Chapter VI looks at the uses for some of the probabilities as they apply to making design level, operational level and tactical level decisions for a single multi-warhead UCAV. Chapter VII is an initial examination of putting the problem in a control type formulation in which we would try to estimate in real time the actual distribution parameters of the environment we are searching and change a control accordingly. Chapter VIII then examines some uses of these probabilities as they apply to a couple of cooperation schemes for search and classification involving two UCAVs. In particular, we evaluate two rules of engagement for each of the cooperative schemes.

We then conclude our work and give some recommendations for future research in Chapter IX.

II. Model Description

Throughout this research, one of three distributions are assumed for the location of targets (T's) and/or false targets (FT's). The most common distribution for the six scenarios is the Poisson distribution.

A random variable whose sample space $S = \{0, 1, 2, ...\}$ has probability mass function $p(\cdot)$ with parameter $\lambda > 0$ given by

$$p(f) = e^{-\lambda} \frac{\lambda^f}{f!} \ , \quad f = 0, 1, 2, ... \tag{2.1}$$

is said to obey the Poisson probability law with parameter λ.

The Poisson field of FT's is characterized by their expected density distribution $\alpha \ [\frac{1}{km^2}]$ so that when a region of area A is searched, the Poisson probability law parameter is $\lambda = \alpha A$.

Equation (2.1) gives the probability of encountering exactly f FT's while searching a Poisson field of FT's. The parameter λ is the expected number of FT encounters occurring over a specific area.

The Poisson field of targets is characterized by their expected density distribution $\beta \ [\frac{1}{km^2}]$ so that when the region A with area A is searched, the Poisson probability law parameter is $\tilde{\lambda} = \beta A$. So then the probability of t T's in A is then

$$p(t) = e^{-\tilde{\lambda}} \frac{\tilde{\lambda}^t}{t!} \ , \quad t = 0, 1, 2, ... \ . \tag{2.2}$$

If the battle space to be searched is region $\mathsf{A_s}$ with area A_s and contains N targets, uniformly distributed, then the probability of a target being in A is

$$P\left\{\text{One Target in } \mathsf{A}\right\} = N\frac{A}{A_s} \ . \tag{2.3}$$

A similar equation is used when dealing with M false targets.

If the location of a target is distributed according to circular normal distribution centered about the origin with a variance σ_T^2, then the probability density for a target at point (x, y) is

$$f(x,y) = \frac{1}{2\pi\sigma_T^2} e^{-\frac{x^2+y^2}{2\sigma_T^2}} \tag{2.4}$$

To find the probability of the target being within a radius r with $r^2 = x^2 + y^2$, we convert to polar coordinates and compute as follows:

$$\begin{aligned} P\{\text{object in } r\} &= \int_0^r \int_0^{2\pi} \frac{1}{2\pi\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} \rho d\theta d\rho && (2.5) \\ &= \int_0^r \frac{1}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} \rho d\rho && (2.6) \\ &= 1 - e^{-\frac{r^2}{2\sigma_T^2}} && (2.7) \end{aligned}$$

The probability of one target out of N targets being in an annulus with inner radius of ρ and width $d\rho$ is

$$N \frac{\rho}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} P_{TR} \tag{2.8}$$

To find the probability of the target being within a radius r of the origin, we simply integrate the annulus probability from 0 to r.

$$\begin{aligned} P(\rho) &= \int_0^r \frac{1}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} \rho d\rho && (2.9) \\ &= 1 - e^{-\frac{r^2}{2\sigma_T^2}}\ . && (2.10) \end{aligned}$$

From this we can see that the probability of the target being outside a circle of radius r is

$$P\{\text{object outside of } r\} = e^{-\frac{r^2}{2\sigma_T^2}} \tag{2.11}$$

Similar equations are developed for M false targets with a variance of σ_{FT}^2.

Irrespective of the assumed distributions, each target or false target can be classified correctly or incorrectly. The *confusion matrix* specifies the probabilities of both correct and incorrect target and false target reports. The basic confusion matrix is shown in Table 2.1, where P_{TR} is the probability of target report and P_{FTR} is the probability of false target report. Specifically, P_{TR} is the probability that a target is reported as such, given that a target is indeed encountered, whereas $1 - P_{TR}$ is the probability that the target is not recognized as a target. Similarly, P_{FTR} is the probability that a false target is correctly classified while $1 - P_{FTR}$ is the probability that the false target is mistaken for a target.

Table 2.1 Simple Confusion Matrix

		Object Encountered	
Object Reported as		T	FT
	T	P_{TR}	$1 - P_{FTR}$
	FT	$1 - P_{TR}$	P_{FTR}

Obviously, the sum of the entries in each column is 1. Ideally, one would like

$$P_{TR} = P_{FTR} = 1 \ ,$$

i.e., one would like the confusion matrix to be the identity matrix. Unfortunately, Autonomous Target Recognition (ATR) is far from achieving this goal and the parameters $0 < P_{TR} < 1$ and $0 < P_{FTR} < 1$ of the confusion matrix play a crucial role in determining the autonomous weapon system's effectiveness. Further, increasing P_{TR} and increasing P_{FTR} are competing objectives in the design of an autonomous target recognition system. The relationship between P_{TR} and the complement of P_{FTR} is directly analogous to the probability of detection and the probability of false alarm as depicted in the classical receiver operating characteristic (ROC) curve from the radar and communication fields. Thus, if one manages to make the P_{TR} parameter to increase, $P_{TR} \rightarrow 1$, at the same time the P_{FTR} parameter will decrease resulting in more false positives. This, coupled with a high density of FT's, can be catastrophic. If multiple types of targets are involved, the confusion matrix is of a higher dimension [27]. In this research we confine our attention to a 2×2 confusion matrices, as illustrated in Table 2.1.

For the remainder of this research, we define the following events:

1. $\mathcal{T}$: The event of a target attack (TA)
2. $\mathcal{T}_{t,(\cdot)}$: The event of t TA's in the space represented by $(\cdot)$ (usually the normalized time, x)
3. $\mathcal{F}$: The event of a false target attack (FTA)
4. $\mathcal{F}_{f,(\cdot)}$: The event of f FTA's in the space represented by $(\cdot)$ (again, usually the normalized time, x)

A FT may potentially fool the ATR algorithm into believing it is a true target. For a single-shot (perishable) munition, the probability of engaging the target in the incremental area ΔA is conditioned on not having engaged a FT prior to arriving at ΔA. Hence, for the single target scenario, the incremental probability of encountering the target in ΔA is

$$\Delta P_E = P\left(\mathcal{F}_{0,A}\right)\frac{\Delta A}{A_s} \ , \tag{2.12}$$

where $\frac{\Delta A}{A_s}$ is the probability that the target is in ΔA and $P\left(\mathcal{F}_{0,A}\right)$ is the probability of no false target attacks (FTA) while searching the region A leading up to ΔA. Previous work [29] has shown that

$$P\left(\mathcal{F}_{0,A}\right) = e^{(1-P_{FTR})\alpha A} \ . \tag{2.13}$$

We confine our development to four scenarios consisting of a rectangular battle space region $\mathsf{A_s}$ of area A_s (see Figure 2.1), and two scenarios using a circular battle space region $\mathsf{A_r}$ of area A_r (see Figure 2.2).

We consider the following six scenarios.

- Scenario 1: A single target (T) is uniformly distributed amongst a Poisson field of false targets (FT).
- Scenario 2: A Poisson field of targets is distributed amongst a Poisson field of false targets.
- Scenario 3: N targets uniformly distributed, and a Poisson field of false targets.

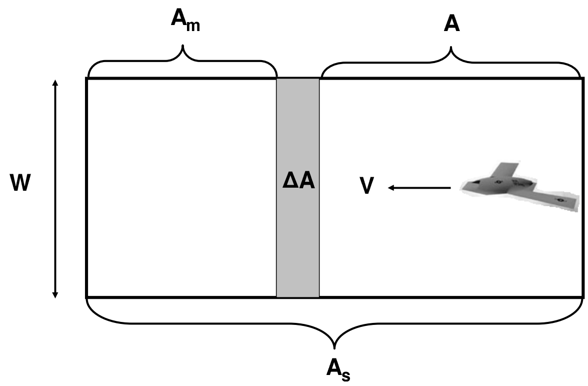

Figure 2.1 Rectangular Battle Space Area Definitions

- Scenario 4: N uniformly distributed targets, and M uniformly distributed false targets.
- Scenario 5: The battle space consists of a circular disc of radius r centered at the origin. There are N targets, distributed according to a circular normal distribution centered at the origin and a Poisson field of false targets.
- Scenario 6: Same battle space as Scenario 5, with N targets distributed according to a circular normal distribution and M false targets distributed according to a circular normal distribution.

For Scenarios 1 thru 4 we assume one Unmanned Combat Aerial Vehicle (UCAV), equipped with w warheads ($w \geq 1$) and flying at a constant speed, V, with a sensor swath width W. Let Υ be the total (deterministic) time required to search region $\mathsf{A_s}$, then $A_s = WV\Upsilon$. Let τ be the time in which the UCAV has searched region A. The area searched, A, is then computed from $A = WV\tau$. Similarly, $\Delta A = WV\Delta\tau$.

For Scenarios 5 and 6 we assume essentially the same thing. Instead of instantly covering a rectangular swath ΔA, we assume we can instantly cover a circular annulus with inner radius ρ and width $\Delta\rho$.

It is convenient to use a normalized time x, $0 \leq x \leq 1$ such that

$$x = \frac{\tau}{\Upsilon} \tag{2.14}$$

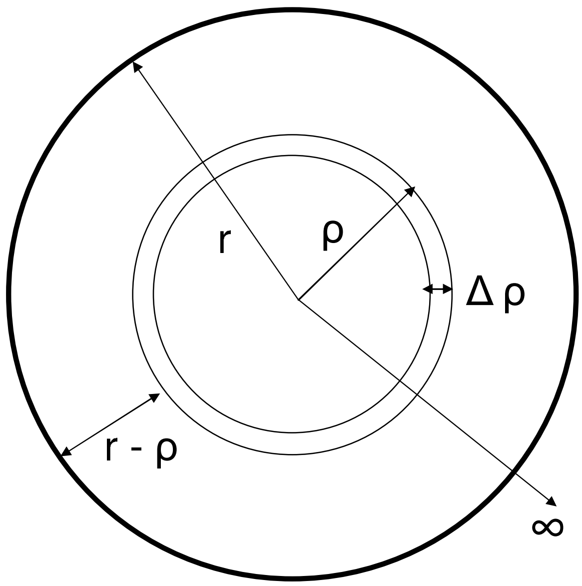

Figure 2.2 Circular Battle Space Area Definitions

and

$$A = A_s x \ . \tag{2.15}$$

Obviously, x can also represent a normalized area.

In addition, we define

$$\lambda_{FT} = \alpha A_s \ . \tag{2.16}$$

The Poisson parameter for false target encounters is given by

$$\lambda = \lambda_{FT} x. \tag{2.17}$$

Similarly, if β is the expected density of targets then

$$\lambda_T = \beta A_s \ , \tag{2.18}$$

and

$$\tilde{\lambda} = \lambda_T x \tag{2.19}$$

is the Poisson parameter for target encounters in the area covered by x.

One objective of this research is to obtain an expression for the probability of occurrence of several important events which can all be defined as various combinations of target attacks (TAs) and/or false target attacks (FTAs). Target attacks have a Poisson parameter $\lambda_{A_T}x$ and false target attacks have Poisson parameter $\lambda_{A_{FT}}x$.

Define

$$\lambda_{A_{FT}} \equiv \alpha_A A_s \ , \tag{2.20}$$

where $\alpha_A = \alpha(1 - P_{FTR})$ and represents the density of FTA situations that would occur in $\mathsf{A_s}$. In view of (2.15) and (2.20), the expected number of FTA's in A is

$$\alpha_A A = \lambda_{A_{FT}} x \ . \tag{2.21}$$

This is the Poisson parameter for FTA's. We show in Appendix B, that developing the equations either way (via FT's or FTA's) is equivalent and to transfer from one method to the other just requires a substitution of variables. That is, we can convert a Poisson field of FT (or T) to a Poisson field of FTA (or TA) as follows:

$$\lambda_{A_{FT}} = (1 - P_{FTR})\lambda_{FT} \ , \tag{2.22}$$

$$\lambda_{A_T} = P_{TR}\lambda_T \ . \tag{2.23}$$

The probabilities of the occurrence of the events of interest can be determined via several methods. In one method, we compute these probabilities by determining the probability that the last attack of interest occurs at a certain time, τ, or in light of equation (2.14), at a certain value of x which represents the percentage of $\mathsf{A_s}$ which has been covered when the attack occurs ($x \in [0, 1]$). Let $p(x)dx$ be defined as the probability of the last attack of interest occurring during the interval $[x, x+dx]$. Calculating the probability that an event occurs in $\mathsf{A_s}$ is just a matter of integrating

$$P(A_s) \quad = \quad \int_0^1 p(x)dx \ . \tag{2.24}$$

The events of interest can be considered as a series of sub-events. These sub-events occur in $\mathsf{A_s}$, A, or $\mathsf{A_s} - \mathsf{A}$ (or $x = 1$, x, or $1 - x$, respectively) . In this method, the probabilities of these sub-events (the 'elemental' probabilities) must be determined to compute the probability of the event of interest. This sequential events method will be covered in Chapter III. Then the probabilities will be calculated using a Markov chain approach. This will be discussed in Chapter V.

III. Sequential Events Method

We now discuss the definition of events and the sub-events of which they are comprised. This formulation is general and does not depend on a particular distribution of targets or false targets. Once the events are defined, we can then compute their probabilities based on the assumed distributions in each scenario. The probabilities of the occurrence of these events are determined in subsequent sections.

3.1 Target Attack

If our UCAV has a single warhead, then for the UCAV to attack a target in ΔA, the UCAV could not have attacked a T or FT in A. The probability of the target attack (TA) occurring in ΔA (i.e. in $[x, x+\Delta x]$) is then

$$P\{\text{TA in } \Delta A\} \quad = \quad P\left(\mathcal{T}_{0,A} \cap \mathcal{F}_{0,A} \cap \mathcal{T}_{1,\Delta A}\right) \ , \tag{3.1}$$

where $\mathcal{T}_{t,X}$ is the event 't TA's in X'. $\mathcal{F}_{f,X}$ is defined as 'f FTA's in X'.

Recall that the area A and the time x have a one-to-one relationship. The use of A in (3.1) is for notational purposes. In all of our calculations of probabilities for the various scenarios, we will develop the equations in terms of the normalized time instead of the area covered. In addition, we shall see that for each of our scenarios, any probability of an attack occurring in ΔA will be some term multiplied by Δx. Therefore, we can express (3.1) in one of two equivalent ways

$$p_1^{(1)}(x)\Delta x = P\left\{\mathcal{T}_{0,A} \cap \mathcal{F}_{0,A} \cap \mathcal{T}_{1,\Delta A}\right\} \ , \tag{3.2}$$

or

$$p_1^{(1)}(x)\Delta x = P\left\{\mathcal{T}_{0,x} \cap \mathcal{F}_{0,x} \cap \mathcal{T}_{1,\Delta x}\right\} \ , \tag{3.3}$$

Regardless of which notation we use, the actual parameter that will be used in the computations will be x. Note we have also introduced the notation $p_{t,f}^{(w)}(x)$ which is the probability of exactly t TA's and f FTA's in A (i.e. in the interval $[0, x]$) assuming w warheads, with $t + f \leq w$. This probability with the second index missing (i.e. $p_t^{(w)}(x)$) represents the probability of exactly t TA's in A regardless of the number of FTA's in A.

The probability of a TA occurring at two different times $(x_1,\, x_2)$ is then

$$
\begin{aligned}
P\left\{\text{TA occurs at } x_1 \text{ or } x_2\right\} \quad = \quad & P\Big\{ \{\mathcal{T}_{0,x_1} \cap \mathcal{F}_{0,x_1} \cap \mathcal{T}_{1,\Delta x_1}\} \cup \\
& \{\mathcal{T}_{0,x_2} \cap \mathcal{F}_{0,x_2} \cap \mathcal{T}_{1,\Delta x_2}\} \Big\} \ . \qquad (3.4)
\end{aligned}
$$

The unioned events are mutually exclusive, therefore we can sum their probabilities. This mutual exclusivity holds true for any set of distinct x's. We can therefore determine the probability of a TA in any range of x's. Determining the probability of a TA in $\mathsf{A_s}$ requires summing (3.2) over all possible A's in A_s (or equivalently, all $x : 0 \leq x \leq 1$). To do this, we will follow the development of the definite integral in Schaum's outline for calculus [5]. We divide the interval [0:1] into n subintervals $h_1, h_2, \ldots, h_n$ by the insertion of $n-1$ points $\xi_1, \xi_2, \ldots, \xi_{n-1}$ where $0 < \xi_1 < \xi_2 < \ldots < \xi_{n-1} < 1$. We denote the length of subinterval h_i by $\Delta_i x = \xi_i - \xi_{i-1}$. On each subinterval, we select a point x_i on the subinterval h_i. Then we have

$$
p_1^{(1)}(x_i)\Delta_i x = P\left\{\mathcal{T}_{0,x_i} \cap \mathcal{F}_{0,x_i} \cap \mathcal{T}_{1,\Delta_i x}\right\} \ . \qquad (3.5)
$$

We then let n approach infinity and sum over all the x_i's in the range in which we are interested.

$$
\begin{aligned}
P_1^{(1)}(A_s) \quad &= \quad \lim_{n \to +\infty} \sum_{i=1}^{n} P\left\{\mathcal{T}_{0,x_i} \cap \mathcal{F}_{0,x_i} \cap \mathcal{T}_{1,\Delta_i x}\right\} \qquad (3.6) \\
&= \quad \lim_{n \to +\infty} \sum_{i=1}^{n} p_1^{(1)}(x_i)\Delta_i x \qquad (3.7)
\end{aligned}
$$

Then using the definition of the definite integral, we have

$$\int_0^1 p_1^{(1)}(x)dx \quad = \quad \lim_{n\to+\infty}\sum_{i=1}^{n} p_1^{(1)}(x_i)\Delta_i x \tag{3.8}$$

$$P_1^{(1)}(A_s) = \int_0^1 p_1^{(1)}(x)dx \ . \tag{3.9}$$

For brevity sake, we will use notation similar to the following notation for future probability derivations;

$$p_1^{(1)}(x)dx \quad = \quad P\left\{\mathcal{T}_{0,x}\cap\mathcal{F}_{0,x}\cap\mathcal{T}_{1,\Delta x}\right\} \ , \tag{3.10}$$

recognizing that several steps are involved as indicated by (3.5) thru (3.8) and that the equality is really only applicable in terms of integration on the left side and infinite series on the right as depicted in (3.8).

With a multiple warhead UCAV, we must ensure we have at least one warhead left after A to attack the T in ΔA, so that

$$p_1^{(w)}(x)dx = P\left(\mathcal{T}_{0,A}\cap\mathcal{F}_{f\le w-1,A}\cap\mathcal{T}_{1,\Delta A}\right) \ , \tag{3.11}$$

where $\mathcal{F}_{f\le w-1,A}$ is the event 'no more than $w-1$ FTA's in A'.

Equation (3.11) is the probability that our last attack of interest (the TA) occurs at the end of A. Or stated equivalently, it is the probability that the first TA occurs at x. Integrating, we obtain the probability of at least one TA assuming w warheads.

$$P_{t\ge1}^{(w)}(A_s) = \int_0^1 p_1^{(w)}(x)dx \ . \tag{3.12}$$

3.2 Probability of an Exact Number of Target and False Target Attacks

For reasons which will become evident, we need to determine $P_{t,f}^{(w)}(A_s)$. Of crucial importance to this discussion is the subtle distinction between encountering a FTA situation and an actual FTA. The same distinction exists between a TA situation and an actual

TA. Since both of our scenarios consist of a Poisson distribution of false targets, we will concentrate on the former.

A FTA situation is one in which a FTA would have occurred if we had a limitless supply of warheads. A FTA situation becomes a FTA if we have a warhead available when we come across that FTA situation. For this probability, we will separate events into two cases. In Case 1, all the warheads are used. In Case 2, not all the warheads are used.

In Case 2, since we have warheads left over, to have exactly f FTA's means we have only come across f FTA situations (or else we would have used more warheads). In Case 1, however, all the warheads have been used and we can come across considerably more FTA situations once we have expended our warheads on the initial f FTA's and t TA's, assuming $t + f = w$. The practical significance of this distinction is that we will have to integrate for Case 1, whereas in Case 2 we will not have to integrate but can determine the probabilities directly for the area in question.

In either case, we must examine two mutually exclusive events. The first event is that in which a TA is the last attack, the second event is that in which a FTA is the last attack:

Case 1 ($t + f = w$): In this case, once the final warhead is released, we do not care if we come across a FTA or TA situation. The probability of occurrence of t TA's and f FTA's is then

$$p_{t,f}^{(t+f=w)}(x)dx = P\begin{pmatrix} \{\mathcal{T}_{t-1,A} \cap \mathcal{F}_{f,A} \cap \mathcal{T}_{1,\Delta A}\} \cup \\ \{\mathcal{T}_{t,A} \cap \mathcal{F}_{f-1,A} \cap \mathcal{F}_{1,\Delta A}\} \end{pmatrix} . \tag{3.13}$$

Since a TA and a FTA are independent of each other and since the unioned events are disjoint, we can break down this equation further;

$$p_{t,f}^{(t+f=w)}(x)dx = P\left(\mathcal{T}_{t-1,A} \cap \mathcal{T}_{1,\Delta A}\right) P\left(\mathcal{F}_{f,A}\right) + P\left(\mathcal{T}_{t,A}\right) P\left(\mathcal{F}_{f-1,A} \cap \mathcal{F}_{1,\Delta A}\right) \tag{3.14}$$

When we integrate (3.14) we then obtain the probability of exactly t TA and f FTA in x,

$$P_{t,f}^{(t+f=w)}(x) = \int_0^x p_{t,f}^{(t+f=w)}(x)dx . \tag{3.15}$$

Note: If we have a situation where $f = 0$ or $t = 0$, we use only the first or last term (respectively) in (3.14).

Case 2 ($t + f < w$): In this case, there can be no attacks after ΔA.

$$p_{t,f}^{(t+f<w)}(x)dx \;=\; P\begin{pmatrix} \{\mathcal{T}_{t-1,A} \cap \mathcal{F}_{f,A} \cap \mathcal{T}_{1,\Delta A} \cap \mathcal{T}_{0,A_s-A} \cap \mathcal{F}_{0,A_s-A}\} \cup \\ \{\mathcal{T}_{t,A} \cap \mathcal{F}_{f-1,A} \cap \mathcal{F}_{1,\Delta A} \cap \mathcal{T}_{0,A_s-A} \cap \mathcal{F}_{0,A_s-A}\} \end{pmatrix}, \quad (3.16)$$

$$P_{t,f}^{(t+f<w)}(A_s) \;=\; \int_0^1 p_{t,f}^{(t+f<w)}(x)dx \; . \quad (3.17)$$

We note that in all six scenarios, the event 'coming across a TA situation' is independent of the event 'coming across a FTA situation'. Also note that the two unioned events in (3.16) are mutually exclusive. Therefore, we can rewrite (3.16) as

$$\begin{aligned} p_{t,f}^{(t+f<w)}(x)dx \;&=\; P\{\mathcal{T}_{t-1,A} \cap \mathcal{T}_{1,\Delta A} \cap \mathcal{T}_{0,A_s-A}\} P\{\mathcal{F}_{f,A} \cap \mathcal{F}_{0,A_s-A}\} + \\ &\quad P\{\mathcal{T}_{t,A} \cap \mathcal{T}_{0,A_s-A}\} P\{\mathcal{F}_{f-1,A} \cap \mathcal{F}_{1,\Delta A} \cap \mathcal{F}_{0,A_s-A}\} \; . \end{aligned} \quad (3.18)$$

Again, if $f = 0$ or $t = 0$, then we only use the first or last term, respectively, in (3.18).

Since there are warheads left after A, then we could not have come across a TA situation, otherwise we would expend another warhead and have an additional TA. Therefore, finding the probability of exactly t TA and f FTA in any given area can be answered directly (without having to integrate). That is, if we can calculate $P\{\mathcal{T}_{t,A}\}$ directly, then we can calculate $P\{\mathcal{T}_{t,A_s}\}$ directly as well. We can determine the probability of some exact number of TA's in $\mathsf{A_s}$. We see that

$$\begin{aligned} P_t^{(w)}(A_s) \;&=\; P\left(\bigcup_{f=0}^{w-t} \{\mathcal{T}_{t,A_s} \cap \mathcal{F}_{f,A_s}\}\right) && (3.19) \\ &=\; \sum_{f=0}^{w-t-1} P_{t,f}^{(t+f<w)}(A_s) + P_{t,f}^{(t+f=w)}(A_s) \; , && (3.20) \end{aligned}$$

where the two terms in (3.20) are defined in (3.17) and (3.15). Also note that in (3.20) and throughout the rest of the research, we adopt the convention that whenever the upper limit on the summation is less than the lower limit, the sum is zero. So then, when $t = w$ we only calculate the last term in (3.20), that is $P_{w,0}^{(t+f=w)}(A_s)$.

We then compute the expected number of TA's as

$$E[t] \quad = \quad \sum_{t=0}^{w} t P_t^{(w)}(A_s)\ . \tag{3.21}$$

Similarly for FTA's we have

$$\begin{aligned} P_{(\cdot),f}^{(w)}(A_s) \quad &= \quad P\left(\bigcup_{t=0}^{w-f} \{\mathcal{T}_{t,A_s} \cap \mathcal{F}_{f,A_s}\}\right) && (3.22) \\ &= \quad \sum_{t=0}^{w-f-1} P_{t,f}^{(t+f<w)}(A_s) + P_{t,f}^{(t+f=w)}(A_s)\ . && (3.23) \end{aligned}$$

We then compute the expected number of FTA's as

$$E[f] \quad = \quad \sum_{f=0}^{w} f P_{(\cdot),f}^{(w)}(A_s)\ . \tag{3.24}$$

3.3 *Probability of a Certain Number of Warheads Remaining After the Region* A *Has Been Searched*

We consider the probability of having w_A warheads remaining after sweeping through a portion of the target area, i.e., we had $w - w_A$ warheads spent in A. The probability of this event is denoted by $P^{w_A}(x)$. We need to consider two cases.

Case 1 ($w_A = 0$): This corresponds to the case in which all warheads are spent in A, i.e., we could have more than w TA and/or FTA situations in A. This is in essence the same as Case 1 in Section 3.2 ($t + f = w$). But instead of integrating from $x = 0$ to $x = 1$, we integrate from 0 to x. We will use the notation A_z to represent the region which grows in size from 0 to A, its area is A.

$$A_z = VW\tau_z \ , \tag{3.25}$$

$$z = \frac{\tau_z}{\Upsilon} \ . \tag{3.26}$$

With these definitions, it is seen that

$$\alpha A_z = \alpha A_s z = \lambda_{FT} z \tag{3.27}$$

$$A - A_z = A_s x - A_s z = A_s(x - z) \tag{3.28}$$

giving

$$p^{w_A=0}(z)dz \quad = \quad P\left(\begin{array}{c} \left\{ \begin{array}{l} \left\{\mathcal{T}_{0,A_z} \cap \mathcal{F}_{(w-1),A_z}\right\} \cup \\ \left\{\mathcal{T}_{1,A_z} \cap \mathcal{F}_{(w-2),A_z}\right\} \cup \cdots \\ \cup \left\{\mathcal{T}_{t,A_z} \cap \mathcal{F}_{(w-1-t),A_z}\right\} \cup \cdots \\ \cup \left\{\mathcal{T}_{(w-1),A_z} \cap \mathcal{F}_{0,A_z}\right\} \end{array} \right\} \cap \\ \{\mathcal{T}_{1,\Delta A_z} \cup \mathcal{F}_{1,\Delta A_z}\} \end{array} \right) \tag{3.29}$$

$$P^{w_A=0}(x) \quad = \quad \int_0^x p^{w_A=0}(z)dz \ . \tag{3.30}$$

Case 2 ($1 \leq w_A \leq w$): This corresponds to the case in which fewer than w warheads are used in A (as in Case 2 in Section 3.2 ($t+f < w$)). Here we also note that to have warheads left over after A means not all the warheads were used by A Therefore, we must not have come across any other attack situations (target or false target) in A, so we are then just looking at the probability of coming across t TA situations and f FTA situations such that $t + f < w$.

$$P^{1\leq w_A \leq w}(x) \quad = \quad P\left(\bigcup_{t=0}^{w-w_A} \{\mathcal{T}_{t,A} \cap \mathcal{F}_{w-w_A-t,A}\} \right) \ ,$$

$$P^{1\leq w_A \leq w}(x) \quad = \quad P\left(\begin{array}{l} \left\{\mathcal{T}_{0,A} \cap \mathcal{F}_{(w-w_A-0),A}\right\} \cup \\ \left\{\mathcal{T}_{1,A} \cap \mathcal{F}_{(w-w_A-1),A}\right\} \cup \cdots \\ \cup \left\{\mathcal{T}_{t,A} \cap \mathcal{F}_{(w-w_A-t),A}\right\} \cup \cdots \\ \cup \left\{\mathcal{T}_{(w-w_A),A} \cap \mathcal{F}_{0,A}\right\} \end{array} \right) \ . \tag{3.31}$$

3.4 *Probability of Additional TA's Given a Certain Number of Warheads Remaining After the Region* A

It is desirable to know the probability of a TA in the rest of the mission given we have w_A warheads remaining. We consider the region $\mathsf{A_y}$ whose area is $A_y = VW\tau_y$ and define $y = \frac{\tau_y}{\Upsilon}$, which goes from 0 to $\mathsf{A_m} = \mathsf{A_s} - \mathsf{A}$. Table 3.1 summarizes the various parameters.

Table 3.1 Summary of Spaces and Variables of Integration

Space	Intermediate Space	Variable of Integration	Lower Limit	Upper Limit
$\mathsf{A_s}$	A	x	0	1
A	$\mathsf{A_z}$	z	0	x
$\mathsf{A_m} = \mathsf{A_s} - \mathsf{A}$	$\mathsf{A_y}$	y	0	$1-x$

We again examine the probability for two cases. When the number of attacks in $\mathsf{A_m}$ are equal to the warheads left (w_A), we have a Case 1 situation. When the number of attacks in $\mathsf{A_m}$ are less than w_A, we have a Case 2 situation. In both cases we are dealing with a conditional probability and will make use of the fact that for the conditional probability of 'A' given 'B', we have

$$P\{A|B\} = \frac{P\{A \cap B\}}{P\{B\}} \ .$$

Case 1 ($t_y + f_y = w_A$): Define $\mathcal{W}_\mathcal{A}$ to be the event 'w_A warheads left after A'. We will also define t_y and f_y to be the number of TA's and FTA's, respectively, in $\mathsf{A_m}$ while t and f are still the number of TA's and FTA's in A. Then

$$P_{t_y}^{(t_y+f_y=w_A)}(y) \quad = \quad \frac{P\left\{\mathcal{T}_{t_y,A_m} \cap \mathcal{F}_{f_y=w_A-t_y,A_m} \cap \mathcal{W}_\mathcal{A}\right\}}{P\{\mathcal{W}_\mathcal{A}\}} \ . \tag{3.32}$$

Recognizing we can have a TA or FTA in ΔA_y, and also recognizing that

$$\mathcal{W}_\mathcal{A} \quad = \quad \bigcup_{t=0}^{w-w_A} \{\mathcal{T}_{t,A} \cap \mathcal{F}_{w-w_A-t,A}\} \ , \tag{3.33}$$

we have

$$p_{t_y}^{(t_y+f_y=w_A)}(y)dy = \frac{P\left\{\left\{\begin{array}{c}\left\{\mathcal{T}_{t_y-1,A_y}\cap\mathcal{F}_{w_A-t_y,A_y}\cap\mathcal{T}_{1,\Delta A_y}\right\}\cup\\ \left\{\mathcal{T}_{t_y,A_y}\cap\mathcal{F}_{w_A-t_y-1,A_y}\cap\mathcal{F}_{1,\Delta A_y}\right\}\end{array}\right\}\cap\{\mathcal{W}_{\mathcal{A}}\}\right\}}{P\{\mathcal{W}_{\mathcal{A}}\}}. \quad (3.34)$$

Note that in all six of our scenarios, the TA's and FTA's are independent of each other, and note that the events separated by the union operator in (3.33) are disjoint. Therefore,

$$p_{t_y}^{(t_y+f_y=w_A)}(y)dy = \frac{\sum_{t=0}^{w-w_A} P\left\{\mathcal{T}_{t_y-1,A_y}\cap\mathcal{T}_{1,\Delta A_y}\cap\mathcal{T}_{t,A}\right\}P\left\{\mathcal{F}_{w_A-t_y,A_y}\cap\mathcal{F}_{w-w_A-t,A}\right\}}{\sum_{t=0}^{w-w_A} P\{\mathcal{T}_{t,A}\}P\{\mathcal{F}_{w-w_A-t,A}\}} + \frac{\sum_{t=0}^{w-w_A} P\left\{\mathcal{T}_{t_y,A_y}\cap\mathcal{T}_{t,A}\right\}P\left\{\mathcal{F}_{w_A-t_y-1,A_y}\cap\mathcal{F}_{1,\Delta A_y}\cap\mathcal{F}_{w-w_A-t,A}\right\}}{\sum_{t=0}^{w-w_A} P\{\mathcal{T}_{t,A}\}P\{\mathcal{F}_{w-w_A-t,A}\}}, \quad (3.35)$$

$$P_{t_y}^{(t_y+f_y=w_A)}(x) = \int_0^{1-x} p_{t_y}^{(t_y+f_y=w_A)}(y)dy \quad (3.36)$$

Case 2 ($t_y + f_y < w_A$): We can only have $t_y + f_y$ attack situations in A_m , i.e. no TA or FTA in $\mathsf{A}_\mathsf{m} - \mathsf{A}_\mathsf{y}$. Recall that for $< w$ attacks, we can calculate the probabilities directly. i.e.

$$P_{t_y,f_y}^{(t_y+f_y<w_A)}(y) = \frac{P\left\{\left\{\mathcal{T}_{t_y,A_y}\cap\mathcal{F}_{f_y,A_y}\right\}\cap\left\{\bigcup_{t=0}^{w-w_A}\{\mathcal{T}_{t,A}\cap\mathcal{F}_{w-w_A-t,A}\}\right\}\right\}}{P\left\{\bigcup_{t=0}^{w-w_A}\{\mathcal{T}_{t,A}\cap\mathcal{F}_{w-w_A-t,A}\}\right\}}. \quad (3.37)$$

Because the TA's and FTA's are independent and the events separated by the union operator are disjoint, we have

$$P_{t_y,f_y}^{(t_y+f_y<w_A)}(y) = \frac{\sum_{t=0}^{w-w_A} P\left\{\mathcal{T}_{t_y,A_y}\cap\mathcal{T}_{t,A}\right\}P\left\{\mathcal{F}_{f_y,A_y}\cap\mathcal{F}_{w-w_A-t,A}\right\}}{\sum_{t=0}^{w-w_A} P\{\mathcal{T}_{t,A}\}P\{\mathcal{F}_{w-w_A-t,A}\}}. \quad (3.38)$$

Therefore,

$$P_{t_y}^{(t_y+f_y<w_A)}(y) = \sum_{f_y=0}^{w_A-t_y-1} \frac{\sum_{t=0}^{w-w_A} P\left\{\mathcal{T}_{t_y,A_y} \cap \mathcal{T}_{t,A}\right\} P\left\{\mathcal{F}_{f_y,A_y} \cap \mathcal{F}_{w-w_A-t,A}\right\}}{\sum_{t=0}^{w-w_A} P\left\{\mathcal{T}_{t,A}\right\} P\left\{\mathcal{F}_{w-w_A-t,A}\right\}} . \tag{3.39}$$

Then we sum the 2 Cases to calculate the probability of t TA in A_{m} given w_A warheads remain after A.

$$p_{t_y}^{w_A}(y) = p_{t_y}^{(t_y+f_y=w_A)}(y) + p_{t_y}^{(t_y+f_y<w_A)}(y) \tag{3.40}$$

$$P_{t_y}^{(w_A)}(x) = P_{t_y}^{(t_y+f_y=w_A)}(x) + P_{t_y}^{(t_y+f_y<w_A)}(x) \tag{3.41}$$

Note: whenever we have a negative subscript (such as we would have in $\mathcal{T}_{t_y-1,A_y}$ when $t_y = 0$) we ignore that event (or we say the probability of that event is 0).

3.5 Mission Success and Expected Time of Mission Success

When dealing with multiple targets, we can allow the commander to define mission success by the number of targets he/she wishes to be attacked. We designate this number as m. We can easily utilize a similar definition of mission success which looks at the number of targets killed instead of the number of targets attacked by multiplying the probabilities in this section by the probability of a target kill given a target attack, P_k.

The probability of mission success is the probability of at least m TA's.

$$p_{t\geq m}^{(w)}(A)\Delta A = P\left\{\mathcal{T}_{m-1,A} \cap \mathcal{F}_{f\leq(w-m),A} \cap \mathcal{T}_{1,\Delta A}\right\} \tag{3.42}$$

$$= P\left\{\bigcup_{f=0}^{w-m} \mathcal{T}_{m-1,A} \cap \mathcal{F}_{f,A} \cap \mathcal{T}_{1,\Delta A}\right\} \tag{3.43}$$

$$p_{t\geq m}^{(w)}(x)\Delta x = P\left\{\bigcup_{f=0}^{w-m} \mathcal{T}_{m-1,x} \cap \mathcal{F}_{f,x} \cap \mathcal{T}_{1,\Delta x}\right\} \tag{3.44}$$

$$\int_0^x p_{t\geq m}^{(w)}(z)dz = \lim_{n\to\infty} \sum_{i=1}^{n} p_{t\geq m}^{(w)}(z_i)\Delta_i z \tag{3.45}$$

$$P_{t\geq m}^{(w)}(A_s) = \int_0^1 p_{t\geq m}^{(w)}(x)dx \tag{3.46}$$

We can determine the expected time of mission success (given a mission success). Given (2.14) we see that calculating the expected time of mission success is equivalent to calculating the expected x of mission success. Let x_s denote the normalized time/space of mission success. Its expected value is $E[x_s]$. To determine this expected value, we normalize the integrand of (3.46) to convert it to a probability density function (pdf), designated as $f^{(w)}_{t\geq m}(x)$, and compute the expected value in the usual way.

$$f^{(w)}_{t\geq m}(x) = \frac{p^{(w)}_{t\geq m}(x)}{P^{(w)}_{t\geq m}(A_s)} \tag{3.47}$$

$$E[x_s] = \int_0^1 x f^{(w)}_{t\geq m}(x)dx = \int_0^1 x \frac{p^{(w)}_{t\geq m}(x)}{P^{(w)}_{t\geq m}(A_s)} dx \tag{3.48}$$

3.6 Mission Failure

We will declare an event 'Mission Failure' if one of two sub-events occur. Either:

1. All warheads have been expended before attacking the mth target or;

2. The UCAV has searched battle space $\mathsf{A_s}$ without attacking the mth target with warheads remaining.

Using our notation

$$P\{\text{mission failure}\} = P^{(w)}_{t<m}(A_s) \ , \tag{3.49}$$

$$= 1 - P^{(w)}_{t\geq m}(A_s) \ , \tag{3.50}$$

$$= P\begin{pmatrix} \{\mathcal{T}_{t<m,A_s} \cap \mathcal{F}_{w-t,A_s}\} \cup \\ \{\mathcal{T}_{t<m,A_s} \cap \mathcal{F}_{f<w-t,A_s}\} \end{pmatrix} \ . \tag{3.51}$$

Utilizing the mutually exclusive nature of the events separated by the union operator, we calculate (3.51) as

$$P^{(w)}_{t<m}(A_s) = \sum_{t=0}^{m-1} P^{(t+f=w)}_{t,w-t}(A_s) + \sum_{t=0}^{m-1} \sum_{f=0}^{w-1-t} P^{(t+f<w)}_{t,f}(A_s) \tag{3.52}$$

where the last two terms are given in (3.15) and (3.17). In addition to determining the probability for mission success and failure, we can also examine the expected useful life of the UCAV.

3.7 Expected Vehicle Longevity

We define vehicle longevity to be the useful life of the UCAV. This usefulness lasts only as long as there is fuel left in the UCAV (or area to be searched) and warheads left to attack targets. More formally, we define expected vehicle longevity as follows:

Vehicle Longevity The time at which the last warhead is expended OR the time at which the UCAV runs out of fuel (or has reached the end of the search area) without expending the last (wth) warhead.

This definition does not depend on attacking a specific number of targets or false targets. It merely depends on the total number of attacks.

The probability of all warheads being used is found in (3.29) and the probability of reaching the end of A_s without expending the wth warhead is found in (3.31). To summarize here, we define W_u to be the random variable representing the number of warheads used. Its realization is w_u, and we note

$$w_u = w - w_A \; . \tag{3.53}$$

The event 'vehicle longevity' will be denoted as $\mathcal{VL}$. The probability of this event is

$$P\{\mathcal{VL}\} \;=\; P\left\{ \bigcup_{w_u=0}^{w} W_u = w_u \right\} \; . \tag{3.54}$$

Each of the unioned events are mutually exclusive therefore we sum their probabilities. In addition, recall the event $\mathcal{VL}$ is composed of two parts. Either all the warheads are used ($W_u = w$) or not ($W_u = w_u : w_u < w$), therefore,

$$P\{\mathcal{VL}\} \;=\; P\{W_u = w\} + P\left\{ \bigcup_{w_u=0}^{w-1} W_u = w_u \right\} \; . \tag{3.55}$$

The unioned events in (3.55) are mutually exclusive, and we can obtain the events descriptions from (3.29) and (3.31), so we can write (3.55) (in terms of A_z and ΔA_z) as

$$\begin{aligned} P\{\mathcal{VL}_{A_z}\} &= P\left\{\bigcup_{t=0}^{w-1}\{\mathcal{T}_{t,A_z}\cap\mathcal{F}_{w-1-t,A_z}\}\cap\{\mathcal{T}_{1,\Delta A_z}\cap\mathcal{F}_{1,\Delta A_z}\}\right\}+ \\ &\sum_{w_u=0}^{w-1}P\left\{\bigcup_{t=0}^{w_u}\{\mathcal{T}_{t,A_z}\cap\mathcal{F}_{w_u-t,A_z}\}\right\}\ . \end{aligned} \tag{3.56}$$

It is readily seen that all the unioned events in (3.56) are mutually exclusive. In addition, we recognize that we can evaluate (3.56) for any value of A_z (i.e. A_{z_i}) and therefore any value of $z_i : 0 \leq z_i \leq 1$. Recall that expected vehicle longevity is essentially composed of two mutually exclusive and exhaustive events. Either the vehicle has expended all its warheads or it has run out of fuel (or reached the end of the battle space).

If the UCAV has lost its usefulness because all warheads were expended (the first term in (3.56)), then the expected vehicle longevity would be the time of the wth attack. However, if the UCAV lost its usefulness because it ran out of fuel (or ran out of assigned search space), then we would not declare the end of its usefulness until the end of A_s (i.e. when $z = 1$). Let x_{vl} denote the vehicle longevity (as a normalized time). Its expected value is $E[x_f]$, and its equation is

$$\begin{aligned} E[x_{vl}] &= \lim_{n\to+\infty}\sum_{i=1}^{n} z_i\sum_{t=0}^{w-1}\Big[P\{\mathcal{T}_{t,z_i}\cap\mathcal{F}_{w-1-t,z_i}\cap\mathcal{T}_{1,\Delta_i z}\}+ \\ &P\{\mathcal{T}_{t,z_i}\cap\mathcal{F}_{w-1-t,z_i}\cap\mathcal{F}_{1,\Delta_i z}\}\Big]+ \\ &(1)\sum_{w_u=0}^{w-1}\sum_{t=0}^{w_u}P\{\mathcal{T}_{t,A_s}\cap\mathcal{F}_{w_u-t,A_s}\}\ . \end{aligned} \tag{3.57}$$

In the next chapter we show how to use these probabilities to compute the various probabilities for the scenarios. We will do this exhaustively for Scenarios 1 and 2 and then just concentrate on the very important $P_{t,f}^{(w)}(x)$ probability for Scenarios 4 thru 6.

IV. *Calculations Using Events Model*

4.1 Scenario 1

The first scenario consists of a single target (T) uniformly distributed amongst a Poisson field of false targets (FT's) in a battle space $\mathsf{A_s}$ with area A_s (see Figure 2.1). The event 'FTA' includes coming across a FT and mistaking it for a T. We simplify our discussion for now by wrapping those events up in the Poisson FTA parameter $\lambda_{A_{FT}}$, which represents the mean number of FTA situations that would occur in $\mathsf{A_s}$.

Since we only have one T in Scenario 1, (3.2) can be rewritten as

$$\begin{aligned} p_1^{(1)}(x)dx &= P\left\{(\mathcal{T}_{1,\Delta A}) \cap (\mathcal{F}_{0,A})\right\} & (4.1) \\ &= P\left\{\mathcal{T}_{1,\Delta A}\right\} P\left\{\mathcal{F}_{0,A}\right\} & (4.2) \end{aligned}$$

since the two events are mutually independent. Examining each probability,

$$P(\mathcal{T}_{1,\Delta A}) = P\left\{(T_{1,\Delta A}) \cap (RT_1)\right\} \tag{4.3}$$

where $T_{1,\Delta A}$ is the event 'one T in ΔA' and 'RT_1' is the event 'one target recognized for what it is'. The event 'RT_1' is conditioned on there being a T in the area of interest. These events are also independent, so we can express $P(\mathcal{T}_{1,\Delta A})$ as the probability of a target in the area element ΔA viz., $\frac{WVdt}{A_s} = dx$, times the probability P_{TR} of a target report. i.e.

$$P(\mathcal{T}_{1,\Delta A}) = P_{TR}dx\ . \tag{4.4}$$

When the UCAV has w ($w \geq 2$) warheads,

$$\begin{aligned} p_1^{(w)}(x)dx &= P\left\{(\mathcal{T}_{1,\Delta A}) \cap (\mathcal{F}_{f\leq w-1,A})\right\} & (4.5) \\ &= \sum_{j=0}^{w-1} e^{-\lambda_{A_{FT}}x} \frac{(\lambda_{A_{FT}}x)^j}{j!} P_{TR}dx & (4.6) \end{aligned}$$

The probability of the target being attacked during the UCAV's battle space sweep is then

$$P_1^{(w)}(A_s) \quad = \quad P_{TR} \sum_{j=0}^{w-1} \frac{(\lambda_{A_{FT}})^j}{(j)!} \int_0^1 e^{-\lambda_{A_{FT}} x} x^j dx \ . \tag{4.7}$$

Equations (4.8) thru (4.16) are the elemental probabilities for Scenario 1:

$$P(\mathcal{F}_{1,\Delta A}) = \alpha_A \Delta A = \alpha_A V W d\tau \quad = \quad \alpha_A \frac{A_s}{\Upsilon} d\tau = \lambda_{A_{FT}} dx \tag{4.8}$$

$$P\left\{\mathcal{T}_{0,A}\right\} \quad = \quad P\left\{T_{0,A} \cup (T_{1,A} \cap NRT_1)\right\} \tag{4.9}$$

$$P\{T_{1,A}\} \quad = \quad x \tag{4.10}$$

$$P\left\{\mathcal{T}_{0,A} \cap \mathcal{T}_{1,\Delta A} \cap \mathcal{T}_{0,A_s - A}\right\} \quad = \quad P_{TR} dx \tag{4.11}$$

$$P\left\{\mathcal{T}_{1,A} \cap \mathcal{T}_{0,A_s - A}\right\} \quad = \quad P_{TR} x \tag{4.12}$$

$$P\left\{\mathcal{F}_{f,A} \cap \mathcal{F}_{0,A_s - A}\right\} \quad = \quad e^{-\lambda_{A_{FT}}} \frac{(\lambda_{A_{FT}} x)^f}{f!} \tag{4.13}$$

$$P\left\{\mathcal{F}_{f-1,A} \cap \mathcal{F}_{1,\Delta A} \cap \mathcal{F}_{0,A_s - A}\right\} \quad = \quad e^{-\lambda_{A_{FT}}} \frac{(\lambda_{A_{FT}} x)^{f-1}}{(f-1)!} \lambda_{A_{FT}} dx \tag{4.14}$$

$$P\{\mathcal{T}_{0,A_z}\} \quad = \quad 1 - P_{TR} z \tag{4.15}$$

$$P\{\mathcal{T}_{0,A_m}\} \quad = \quad 1 - (1-x) P_{TR} \tag{4.16}$$

where NRT_1 represents the event 'target not recognized for what it is'.

4.1.1 Probability of an Exact Number of Target and False Target Attacks. The probability of $P_{t,f}^{(w)}(A_s)$ is determined from (3.15) and (3.17 and each case has 2 subcases ($t = 0,\ t = 1$).

Case 1: $t + f = w$

Subcase 1: $t = 0 \Rightarrow f = w$. In this case, we only want $f - 1$ FTA's and no TA's prior to the final warhead;

$$P_{0,f=w}^{(w)}(A_s) \quad = \quad \frac{\lambda_{A_{FT}}^w}{(w-1)!} \left(\int_0^1 e^{-\lambda_{A_{FT}} x} x^{w-1} dx - P_{TR} \int_0^1 e^{-\lambda_{A_{FT}} x} x^w dx \right) \ . \tag{4.17}$$

We can also put this in terms of the incomplete gamma function. It is defined as

$$\gamma(\alpha, z) = \int_0^z e^{-\zeta}\zeta^{\alpha-1}d\zeta \ . \tag{4.18}$$

This means

$$\int_0^x e^{\lambda\xi}\xi^{\alpha-1}d\xi \quad = \quad \frac{\gamma(\alpha, \lambda x)}{\lambda^\alpha} \ . \tag{4.19}$$

We note the gamma function itself is

$$\Gamma(n+1) = n! \ . \tag{4.20}$$

When we divide $\gamma(\alpha, z)$ by $\Gamma(\alpha)$, we have what some have called the regularized incomplete gamma function.

When using the incomplete gamma function, (4.17) becomes

$$P_{0,f=w}^{(t+f=w)}(A_s) \quad = \quad \frac{\gamma(w, \lambda_{A_{FT}})}{\Gamma(w)} - \frac{P_{TR}w}{\lambda_{A_{FT}}}\frac{\gamma(w+1, \lambda_{A_{FT}})}{\Gamma(w+1)} \ . \tag{4.21}$$

We note here that instead of integrating from 0 to 1, we can integrate from 0 to x, making this probability a function of x.

$$P_{0,f=w}^{(t+f=w)}(x) \quad = \quad \frac{\gamma(w, \lambda_{A_{FT}}x)}{\Gamma(w)} - \frac{P_{TR}w}{\lambda_{A_{FT}}}\frac{\gamma(w+1, \lambda_{A_{FT}}x)}{\Gamma(w+1)} \ . \tag{4.22}$$

At times, this form is more useful. We can easily determine the probability over the whole battle space, using this form, by setting $x = 1$.

Subcase 2: $t = 1 \Rightarrow f = w - 1$ Now we use both terms of (3.14).

$$P_{1,f}^{(w)}(A_s) \quad = \quad P_{TR}\lambda_{A_{FT}}^{w-1}\frac{w}{(w-1)!}\int_0^1 \left(e^{-\lambda_{A_{FT}}x}x^{w-1}\right)dx \ , \tag{4.23}$$

$$= \quad \frac{wP_{TR}}{\lambda_{A_{FT}}}\frac{\gamma(w, \lambda_{A_{FT}})}{\Gamma(w)} \tag{4.24}$$

Case 2: $t + f < w$ When $t = 1$ integrating (3.18) gives

$$
\begin{aligned}
P_{1,f}^{(t+f<w)}(A_s) &= \int_0^1 \left(P_{TR} dx e^{-\lambda_{A_{FT}}} \frac{(\lambda_{A_{FT}} x)^f}{f!} + \right. \\
&\qquad \left. P_{TR} x e^{-\lambda_{A_{FT}}} \frac{(\lambda_{A_{FT}} x)^{f-1}}{(f-1)!} \lambda_{A_{FT}} \right) dx && (4.25) \\
&= P_{TR} e^{-\lambda_{A_{FT}}} \lambda_{A_{FT}}^f \left(\frac{f+1}{f!} \right) \int_0^1 x^f dx && (4.26) \\
&= P_{TR} e^{-\lambda_{A_{FT}}} \lambda_{A_{FT}}^f \left(\frac{f+1}{f!} \right) \frac{1}{f+1} && (4.27) \\
&= P_{TR} e^{-\lambda_{A_{FT}}} \frac{(\lambda_{A_{FT}})^f}{f!} \ , && (4.28)
\end{aligned}
$$

which is equivalent to computing $P\{\mathcal{T}_{1,A_s} \cap \mathcal{F}_{f,A_s}\}$ directly. In other words, when $t+f < w$, we are seeking the case of t TA situations and f FTA situations. When $t = 0$, we have

$$
\begin{aligned}
P_{0,f}^{(t+f<w)}(A_s) &= (1 - P_{TR}) e^{-\lambda_{A_{FT}}} \int_0^1 \frac{(\lambda_{A_{FT}} x)^{f-1}}{(f-1)!} \lambda_{A_{FT}} dx && (4.29) \\
&= (1 - P_{TR}) e^{-\lambda_{A_{FT}}} \lambda_{A_{FT}}^f \frac{1}{(f-1)!} \int_0^1 x^{f-1} dx && (4.30) \\
&= (1 - P_{TR}) e^{-\lambda_{A_{FT}}} \frac{(\lambda_{A_{FT}})^f}{f!} \ . && (4.31)
\end{aligned}
$$

Moreover, we can see this is also the case for a Poisson distribution of TA and Poisson distribution of FTA, as will be discussed in Section 4.2.

4.1.2 Probability of Specified Number of Target Attacks. It is easy to see that (3.20) becomes one of two equations; either

$$
\begin{aligned}
P_0^{(w)}(A_s) = \ & \frac{\gamma(w, \lambda_{A_{FT}})}{\Gamma(w)} - \frac{P_{TR} w}{\lambda_{A_{FT}}} \frac{\gamma(w+1, \lambda_{A_{FT}})}{\Gamma(w+1)} + \\
& \sum_{f=0}^{w-1} (1 - P_{TR}) e^{-\lambda_{A_{FT}}} \frac{(\lambda_{A_{FT}})^f}{f!}
\end{aligned} \qquad (4.32)
$$

or

$$
P_1^{(w)}(A_s) = \frac{w P_{TR}}{\lambda_{A_{FT}}} \frac{\gamma(w, \lambda_{A_{FT}})}{\Gamma(w)} + \sum_{f=0}^{w-2} (P_{TR}) e^{-\lambda_{A_{FT}}} \frac{(\lambda_{A_{FT}})^f}{f!} \ . \qquad (4.33)
$$

Equations (4.32) and (4.33) should sum to unity (the events they represent are exhaustive). It is easy to show via integration by parts that the sum corresponds to the probability of at most w FTA's in $\mathsf{A_s}$, and this probability is one.

4.1.3 Probability of Specified Number of False Target Attacks. We can also find the probability of a specified number of false target attacks using (3.23). Since the number of targets is less than the number of warheads, we must look at several situations to evaluate (3.23).

Situation 1, $f \leq w-2$: In this situation we will never need $P_{t,f}^{(t+f=w)}(A_s)$. Therefore

$$\begin{aligned} P_{(\cdot),f\leq w-2}^{(w)}(A_s) &= P_{0,f}^{(t+f<w)}(A_s) + P_{1,f}^{(t+f<w)}(A_s) \ , & (4.34) \\ &= (1-P_{TR})e^{-\lambda_{A_{FT}}}\frac{(\lambda_{A_{FT}})^f}{f!} + P_{TR}e^{-\lambda_{A_{FT}}}\frac{(\lambda_{A_{FT}})^f}{f!} \ , & (4.35) \\ &= e^{-\lambda_{A_{FT}}}\frac{(\lambda_{A_{FT}})^f}{f!} \ . & (4.36) \end{aligned}$$

Situation 2, $f = w-1$:

$$\begin{aligned} P_{(\cdot),f=w-1}^{(w)}(A_s) &= P_{0,w-1}^{(t+f<w)}(A_s) + P_{1,w-1}^{(t+f=w)}(A_s) \ , & (4.37) \\ &= (1-P_{TR})e^{-\lambda_{A_{FT}}}\frac{(\lambda_{A_{FT}})^{w-1}}{(w-1)!} + \frac{wP_{TR}}{\lambda_{A_{FT}}}\frac{\gamma(w,\lambda_{A_{FT}})}{\Gamma(w)} \ . & (4.38) \end{aligned}$$

Situation 3, $f = w$:

$$\begin{aligned} P_{(\cdot),f=w}^{(w)}(A_s) &= P_{0,w}^{(t+f=w)}(A_s) \ , & (4.39) \\ &= \frac{\gamma(w,\lambda_{A_{FT}})}{\Gamma(w)} - \frac{wP_{TR}}{\lambda_{A_{FT}}}\frac{\gamma(w+1,\lambda_{A_{FT}})}{\Gamma(w+1)} \ . & (4.40) \end{aligned}$$

So then the expected number of FTA, $E[f]$ is

$$\begin{aligned}
E[f] &= \sum_{f=0}^{w} f P^{(w)}_{(\cdot),f}(A_s) \ , && (4.41)\\
&= \sum_{f=0}^{w-2} f P^{(w)}_{(\cdot),f}(A_s) + (w-1) P^{(w)}_{(\cdot),w-1}(A_s) + w P^{(w)}_{(\cdot),w}(A_s) \ , && (4.42)\\
&= \sum_{f=0}^{w-2} f e^{-\lambda_{A_{FT}}} \frac{(\lambda_{A_{FT}})^f}{f!} + \\
&\quad (w-1)\left((1-P_{TR}x) e^{-\lambda_{A_{FT}}} \frac{(\lambda_{A_{FT}})^{w-1}}{(w-1)!} + \frac{wP_{TR}}{\lambda_{A_{FT}}} \frac{\gamma(w,\lambda_{A_{FT}})}{\Gamma(w)}\right) + \\
&\quad w\left(\frac{\gamma(w,\lambda_{A_{FT}})}{\Gamma(w)} - \frac{wP_{TR}}{\lambda_{A_{FT}}} \frac{\gamma(w+1,\lambda_{A_{FT}})}{\Gamma(w+1)}\right) \ . && (4.43)
\end{aligned}$$

4.1.4 Probability of a Certain Number of Warheads Remaining After Region A *Has Been Searched.* Since the events in (3.29) are mutually exclusive, their probabilities are additive. When all the warheads are used, (3.30) becomes

$$\begin{aligned}
P^{w_A=0}(x) &= \int_0^x P_{TR} e^{-\lambda_{A_{FT}} z} \frac{(\lambda_{A_{FT}} z)^{w-1}}{(w-1)!} dz + \\
&\quad \int_0^x (1-P_{TR}z) e^{-\lambda_{A_{FT}} z} \frac{(\lambda_{A_{FT}} z)^{w-1}}{(w-1)!} \lambda_{A_{FT}} dz + \\
&\quad \int_0^x P_{TR} z e^{-\lambda_{A_{FT}} z} \frac{(\lambda_{A_{FT}} z)^{w-2}}{(w-2)!} \lambda_{A_{FT}} dz \\
&= \frac{P_{TR}\lambda_{A_{FT}}^{w-1} w + \lambda_{A_{FT}}^{w}}{(w-1)!} \int_0^x e^{-\lambda_{A_{FT}} z} z^{w-1} dz - \\
&\quad P_{TR} \frac{\lambda_{A_{FT}}^{w}}{(w-1)!} \int_0^x e^{-\lambda_{A_{FT}} z} z^{w} dz \\
&= \left(\frac{wP_{TR}}{\lambda_{A_{FT}}} + 1\right) \frac{\gamma(w,\lambda_{A_{FT}}x)}{\Gamma(w)} - \frac{wP_{TR}}{\lambda_{A_{FT}}} \frac{\gamma(w+1,\lambda_{A_{FT}}x)}{\Gamma(w+1)} && (4.44)
\end{aligned}$$

When fewer than w warheads are expended in A, (3.31) becomes

$$\begin{aligned}
P^{1\le w_A \le w}(x) &= (1-P_{TR}x) e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^{w-w_A}}{(w-w_A)!} + \\
&\quad P_{TR} x e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^{w-w_A-1}}{(w-w_A-1)!} && (4.45)
\end{aligned}$$

where $\frac{1}{a!} \equiv 0$ if $a < 0$.

4.1.5 Probability of TA Given a Certain Number of Warheads Remaining After the Region A. We first note that $\alpha A_y = \lambda_{A_{FT}} y$. Then (3.36), (3.39) and (3.41) become

$$P_1^{t_y+f_y=w_A}(x) = \frac{w_A P_{TR}}{(1-P_{TR}x)\,\lambda_{A_{FT}} + P_{TR}\,(w-w_A)} \frac{\gamma(w_A, \lambda_{A_{FT}}(1-x))}{\Gamma(w_A)} \tag{4.46}$$

$$P_1^{(t_y+f_y<w_A)}(x) = \sum_{f_y=0}^{w_A-2} \frac{P_{TR}\lambda_{A_{FT}}}{(1-P_{TR}x)\,\lambda_{A_{FT}} + P_{TR}\,(w-w_A)} \times (1-x)e^{-\lambda_{A_{FT}}(1-x)} \frac{(\lambda_{A_{FT}}(1-x))^{f_y}}{f_y!} \tag{4.47}$$

$$P_1^{(w_A)}(x) = \frac{P_{TR}}{(1-P_{TR}x)\,\lambda_{A_{FT}} + P_{TR}\,(w-w_A)} \times \left[\sum_{f_y=0}^{w_A-2} e^{-\lambda_{A_{FT}}(1-x)} \frac{(\lambda_{A_{FT}}(1-x))^{f_y+1}}{f_y!} + w_A \frac{\gamma(w_A, \lambda_{A_{FT}}(1-x))}{\Gamma(w_A)}\right] \tag{4.48}$$

We can also determine the probability of no TA in $\mathsf{A_m}$. For $t = 0$, (3.41) becomes

$$P_0^{w_A}(x) = \sum_{f_y=0}^{w_A-1} \left(1 - \frac{P_{TR}\lambda_{A_{FT}}(1-x)}{(1-P_{TR}x)\,\lambda_{A_{FT}} + P_{TR}\,(w-w_A)}\right) e^{-\lambda_{A_{FT}}(1-x)} \frac{(\lambda_{A_{FT}}(1-x))^{f_y}}{f_y!} + - \frac{P_{TR}w_A}{(1-P_{TR}x)\,\lambda_{A_{FT}} + P_{TR}\,(w-w_A)} \frac{\gamma(w_A+1, \lambda_{A_{FT}}(1-x))}{\Gamma(w_A+1)} + \frac{\gamma(w_A, \lambda_{A_{FT}}(1-x))}{\Gamma(w_A)}\ . \tag{4.49}$$

Equations (4.48) and (4.49) can be shown numerically to sum to unity as expected.

4.1.6 P(Mission Success) and Expected Time of Mission Success. For Scenario 1, (3.46) and (3.48) become

$$P_1^{(w)}(x) = \int_0^x \sum_{f=0}^{w-1} e^{-\lambda_{A_{FT}} z} \frac{(\lambda_{A_{FT}} z)^f}{f!} z^f P_{TR} dz \tag{4.50}$$

$$= \sum_{f=0}^{w-1} \frac{P_{TR}}{\lambda_{A_{FT}}} \frac{\gamma(f+1, \lambda_{A_{FT}} x)}{\Gamma(f+1)} \tag{4.51}$$

$$E[x_s] = \frac{\sum_{f=0}^{w-1} \frac{P_{TR}(f+1)}{\lambda_{A_{FT}}^2} \frac{\gamma(f+2, \lambda_{A_{FT}})}{\Gamma(f+2)}}{P_1^{(w)}(A_s)} \tag{4.52}$$

Note that the P_{TR}'s would cancel, leaving no dependence on P_{TR} in $E[x_s]$.

4.1.7 P(Mission Failure) and Expected Time of Mission Failure. We have already calculated the probability of mission failure $\left(P_0^{(w)}(A_s) \text{ in } (4.32)\right)$.

4.1.8 Expected Vehicle Longevity. For Scenario 1 (3.57) becomes

$$\begin{aligned} E[x_{vl}] = {} & \lim_{n \to +\infty} \sum_{i=1}^{n} z_i \Bigg[P_{TR} \Delta_i z e^{-\lambda_{A_{FT}} z_i} \frac{(\lambda_{A_{FT}} z_i)^{w-1}}{(w-1)!} + \\ & (1 - P_{TR} z_i) e^{-\lambda_{A_{FT}} z_i} \frac{(\lambda_{A_{FT}} z_i)^{w-1}}{(w-1)!} \lambda_{A_{FT}} \Delta_i z + \\ & P_{TR} z_i e^{-\lambda_{A_{FT}} z_i} \frac{(\lambda_{A_{FT}} z_i)^{w-2}}{(w-2)!} \lambda_{A_{FT}} \Delta_i z \Bigg] + \\ & (1) \sum_{w_u=0}^{w-1} \left[(1 - P_{TR}) e^{-\lambda_{A_{FT}}} \frac{(\lambda_{A_{FT}})^{w_u}}{w_u!} + P_{TR} e^{-\lambda_{A_{FT}}} \frac{(\lambda_{A_{FT}})^{w_u - 1}}{(w_u - 1)!} \right] \end{aligned} \tag{4.53}$$

recalling that if $w_u - 1 < 0$ then we say the term associated with $\frac{1}{(w_u-1)!} = 0$.

Utilizing the definition of the definite integral, we have

$$
\begin{aligned}
E[x_{vl}] &= \lim_{n\to+\infty}\int_0^1 z\left[P_{TR}dze^{-\lambda_{A_{FT}}z}\frac{(\lambda_{A_{FT}}z)^{w-1}}{(w-1)!}+\right. \\
&\quad (1-P_{TR}z)e^{-\lambda_{A_{FT}}z}\frac{(\lambda_{A_{FT}}z)^{w-1}}{(w-1)!}\lambda_{A_{FT}}dz+ \\
&\quad \left.P_{TR}ze^{-\lambda_{A_{FT}}z}\frac{(\lambda_{A_{FT}}z)^{w-2}}{(w-2)!}\lambda_{A_{FT}}dz\right]+ \\
&\quad (1)\sum_{w_u=0}^{w-1}\left[(1-P_{TR})e^{-\lambda_{A_{FT}}}\frac{(\lambda_{A_{FT}})^{w_u}}{w_u!}+P_{TR}e^{-\lambda_{A_{FT}}}\frac{(\lambda_{A_{FT}})^{w_u-1}}{(w_u-1)!}\right] \qquad (4.54) \\
&= \left[w\frac{P_{TR}\lambda_{A_{FT}}^{w-1}w+\lambda_{A_{FT}}^{w}}{\lambda_{A_{FT}}^{w+1}}\frac{\gamma(w+1,\lambda_{A_{FT}})}{\Gamma(w+1)}-\frac{P_{TR}w(w+1)}{\lambda_{A_{FT}}^2}\frac{\gamma(w+2,\lambda_{A_{FT}})}{\Gamma(w+2)}\right]+ \\
&\quad \sum_{w_u=0}^{w-1}\left[(1-P_{TR})e^{-\lambda_{A_{FT}}}\frac{(\lambda_{A_{FT}})^{w_u}}{w_u!}+P_{TR}e^{-\lambda_{A_{FT}}}\frac{(\lambda_{A_{FT}})^{w_u-1}}{(w_u-1)!}\right]\ . \qquad (4.55)
\end{aligned}
$$

4.2 Scenario 2

Scenario 2 consists of a Poisson field of targets and a Poisson field of false targets. For this new distribution we have the parameters β to represent the density of T's in the region $\mathsf{A_s}$, β_A represents the density of TA's in $\mathsf{A_s}$. We then define

$$
\begin{aligned}
\lambda_T &= \beta A_s\ , &(4.56)\\
\Rightarrow \beta A &= \lambda_T x\ . &(4.57)\\
\lambda_{A_T} &= \beta_A A_s\ , &(4.58)\\
\Rightarrow \beta_A A &= \lambda_{A_T} x\ . &(4.59)
\end{aligned}
$$

Thus we have

$$P\{\mathcal{T}_{t,A}\} = e^{-\lambda_{A_T}x}\frac{(\lambda_{A_T}x)^t}{t!}\ , \qquad (4.60)$$

$$P\{\mathcal{T}_{t,\Delta A}\} = \lambda_{A_T}dx\ , \qquad (4.61)$$

where $P\{\mathcal{T}_{t,\Delta A}\}$ is a differential probability. In addition, since Scenario 2 involves a possibly infinite number of TA situations, the joint probabilities involving TA are independent (as the FTA's were in Scenario 1), i.e.,

$$P\left\{\mathcal{T}_{0,A} \cap \mathcal{T}_{1,\Delta A}\right\} = e^{-\lambda_{A_T} x}\lambda_{A_T} dx \ . \tag{4.62}$$

We will continue to formulate the probabilities in terms of a Poisson distribution of TA's and FTA's although (as seen in Scenario 1) we can easily make the following substitutions to generalize to Poisson distribution of T and FT with the associated confusion matrix.

$$\lambda_{A_{FT}} = \lambda_{FT}(1 - P_{FTR}) \ , \tag{4.63}$$

$$\lambda_{A_T} = \lambda_T P_{TR} \ . \tag{4.64}$$

For the single warhead case ($w = 1$) in Scenario 2, (3.9) becomes

$$P_1^{(1)}(A_s) \ = \ \int_0^1 e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x}\lambda_{A_T} dx \ . \tag{4.65}$$

For multiple warheads in Scenario 2, (3.12) (the probability of at least one TA in $\mathsf{A_s}$) becomes

$$P_{t\geq 1}^{(w)}(A_s) \ = \ \int_0^1 \lambda_{A_T} e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x} \sum_{f=0}^{w-1} \frac{(\lambda_{A_{FT}} x)^f}{f!} dx \ . \tag{4.66}$$

The FTA elemental probabilities for Scenario 2 were determined in Scenario 1. The probabilities are similar for the TA elemental probabilities.

$$P(\mathcal{T}_{t,A_y}) \ = \ e^{-\lambda_{A_T} y}\frac{(\lambda_{A_T} y)^t}{t!} \ , \tag{4.67}$$

$$P(\mathcal{T}_{1,\Delta A}) \ = \ \lambda_{A_T} dy \ , \tag{4.68}$$

$$P(\mathcal{T}_{0,A_m - A_y}) \ = \ e^{-\lambda_{A_T}(1-x-y)} \ , \tag{4.69}$$

$$P(\mathcal{F}_{f\leq w_A - t, A_y}) \ = \ \sum_{f=0}^{w_A - t} e^{-\lambda_{A_{FT}} y}\frac{(\lambda_{A_{FT}} y)^f}{f!} \ . \tag{4.70}$$

4.2.1 Probability of an Exact Number of Target and False Target Attacks. Since we are now dealing with a distribution of TA's and FTA's vice a distribution of T's and

FT's, the confusion matrix is absorbed by the attack distribution parameters found in (4.63) and (4.64). Therefore, in Scenario 2 we do not have subcases as in Scenario 1.

Case 1 [$t+f=w$ $(t \geq 0)$]: Equation (3.15) becomes

$$P_{t,w-t}^{(w=t+f)}(A_s) \quad = \quad \lambda_{A_T}^{t}\lambda_{A_{FT}}^{w-t}\frac{w}{(t)!(w-t)!}\int_0^1 \left\{e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x}x^{w-1}\right\}dx \tag{4.71}$$

After converting to incomplete gamma notation, we obtain

$$P_{t,w-t}^{(w=t+f)}(A_s) = \lambda_{A_T}^{t}\lambda_{A_{FT}}^{w-t}\binom{w}{t}\frac{\gamma(w,(\lambda_{A_T}+\lambda_{A_{FT}}))}{\Gamma(w)(\lambda_{A_T}+\lambda_{A_{FT}})^w} \tag{4.72}$$

Case 2 [$t+f<w$ $(t \geq 0)$]: After some simplification of (3.17), we see that

$$P_{t,f}^{(w>t+f)}(A_s) = e^{-(\lambda_{A_T}+\lambda_{A_{FT}})}\lambda_{A_T}^{t}\lambda_{A_{FT}}^{f}\frac{1}{(t)!(f)!} \tag{4.73}$$

$$p_{t,f}^{(w>t+f)}(x) = \left\{e^{-(\lambda_{A_T}+\lambda_{A_{FT}})}\lambda_{A_T}^{t}\lambda_{A_{FT}}^{f}\frac{t+f}{(t)!(f)!}x^{t+f-1}\right\} \tag{4.74}$$

4.2.2 Probability of Specified Number of Target Attacks and Expected Number. Substituting (4.71) and (4.73) into (3.20) we calculate the overall probability of t TA's.

$$\begin{aligned} P_t^{(w)}(A_s) = \quad & \lambda_{A_T}^{t}\lambda_{A_{FT}}^{w-t}\frac{w}{t!(w-t)!}\int_0^1 \left\{e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x}x^{w-1}\right\}dx + \\ & \sum_{f=0}^{w-t-1} e^{-(\lambda_{A_T}+\lambda_{A_{FT}})}\lambda_{A_T}^{t}\lambda_{A_{FT}}^{f}\frac{1}{t!f!} \end{aligned} \tag{4.75}$$

Now, it is easy to show that $\sum_{t=0}^{w} P_t^{(w)}(A_s) = 1$ as expected, since having up to w TA's in $\mathsf{A_s}$ is exhaustive.

So then the expected number of TA, $E[t]$ is

$$E[t] \quad = \quad \sum_{t=0}^{w} tP_t^{(w)}(A_s)\ , \tag{4.76}$$

(4.77)

Recall our notation that whenever the upper limit on the summation sign is less than zero, then the sum is zero.

$$
\begin{aligned}
E[t] \quad = \quad & \sum_{t=0}^{w} t \frac{\lambda_{A_T}^{t} \lambda_{A_{FT}}^{(w-t)}}{(\lambda_{A_T} + \lambda_{A_{FT}})^{w}} \binom{w}{t} \frac{\gamma(w, (\lambda_{A_T} + \lambda_{A_{FT}})\, x)}{\Gamma(w)} + \\
& \sum_{t=0}^{w-1} \sum_{f=0}^{w-t-1} t \frac{(\lambda_{A_T} x)^{t}}{t!} \frac{(\lambda_{A_{FT}} x)^{f}}{f!} e^{-(\lambda_{A_T} + \lambda_{A_{FT}})x} \ . \qquad (4.78)
\end{aligned}
$$

4.2.3 Probability of Specified Number of False Target Attacks and Expected Number. Similarly, for false target attacks we have

$$
\begin{aligned}
P_{(\cdot),f}^{(w)}(A_s) \quad = \quad & \frac{\lambda_{A_T}^{w-f} \lambda_{A_{FT}}^{f}}{(\lambda_{A_T} + \lambda_{A_{FT}})^{w}} \binom{w}{f} \frac{\gamma(w, (\lambda_{A_T} + \lambda_{A_{FT}})\, x)}{\Gamma(w)} + \\
& \sum_{t=0}^{w-f-1} \frac{(\lambda_{A_T} x)^{t}}{t!} \frac{(\lambda_{A_{FT}} x)^{f}}{f!} e^{-(\lambda_{A_T} + \lambda_{A_{FT}})x} \ . \qquad (4.79)
\end{aligned}
$$

So then

$$
\begin{aligned}
E[f] \quad = \quad & \sum_{f=0}^{w} f \frac{\lambda_{A_T}^{w-f} \lambda_{A_{FT}}^{f}}{(\lambda_{A_T} + \lambda_{A_{FT}})^{w}} \binom{w}{f} \frac{\gamma(w, (\lambda_{A_T} + \lambda_{A_{FT}})\, x)}{\Gamma(w)} + \\
& \sum_{f=0}^{w-1} \sum_{t=0}^{w-f-1} f \frac{(\lambda_{A_T} x)^{t}}{t!} \frac{(\lambda_{A_{FT}} x)^{f}}{f!} e^{-(\lambda_{A_T} + \lambda_{A_{FT}})x} \ . \qquad (4.80)
\end{aligned}
$$

4.2.4 Probability of a Certain Number of Warheads Remaining After Region A Has Been Searched. Application of (3.30) and (3.31) to Scenario 2 is straightforward. We compute

$$
\begin{aligned}
P^{w_A=0}(x) = \quad & (\lambda_{A_T} + \lambda_{A_{FT}}) \left(\int_0^x e^{-(\lambda_{A_T} + \lambda_{A_{FT}})z} z^{w-1} dz \right) \\
& \times \left(\sum_{t=0}^{w-1} \frac{(\lambda_{A_T})^{t}}{t!} \frac{(\lambda_{A_{FT}})^{w-1-t}}{(w-1-t)!} \right) \qquad (4.81)
\end{aligned}
$$

$$= (\lambda_{A_T} + \lambda_{A_{FT}}) \frac{\gamma(w, (\lambda_{A_T} + \lambda_{A_{FT}})x)}{(\lambda_{A_T} + \lambda_{A_{FT}})^w} \left(\sum_{t=0}^{w-1} \binom{w-1}{t} \frac{\lambda_{A_T}^t \lambda_{A_{FT}}^{w-1-t}}{(w-1)!} \right) \tag{4.82}$$

$$= \frac{\gamma(w, (\lambda_{A_T} + \lambda_{A_{FT}})x)}{\Gamma(w)} \tag{4.83}$$

$$P^{1 \le w_A \le w}(x) = e^{-(\lambda_{A_T} + \lambda_{A_{FT}})x} x^{w-w_A} \sum_{t=0}^{w-w_A} \frac{(\lambda_{A_T})^t}{t!} \frac{(\lambda_{A_{FT}})^{w-w_A-t}}{(w-w_A-t)!} , \tag{4.84}$$

$$= e^{-(\lambda_{A_T} + \lambda_{A_{FT}})x} \frac{((\lambda_{A_T} + \lambda_{A_{FT}})x)^{w-w_A}}{(w-w_A)!} . \tag{4.85}$$

To illustrate the results of (4.82) and (4.85) we look at a sample case where $n = 3$, $\lambda_{A_T} = 3$, and $\lambda_{A_{FT}} = 10$. We sum the previous probabilities and we see from Figure 4.1 that they sum to one. We also see in this plot that the probability of having all the warheads available is one at $x = 0$ and quickly approaches zero. Since we are expecting an average of three target attack situations and ten false target attack situations in the battle space ($\lambda_{A_T} = 3, \lambda_{A_{FT}} = 10$), the probability of having no warheads left becomes a practical certainty as we approach the end of the battle space. Also note that since we are looking at the probability of an exact number of warheads (which implies an exact number of attacks - no more, no less), the probability for an intermediate number of warheads peaks at some point in the battle space. For example, for $w_A = 2$ (number of attacks = 1), we initially have a probability of zero at $x = 0$ but at $x \simeq .07$ the probability peaks. This also coincides with the most likely place where we would have our 2nd attack. After that, it becomes more likely that another attack will occur (whether a TA or FTA).

A plot of the expected value of w_A for the same set of parameters is found in Figure 4.2. Since the probabilities in (4.82) and (4.85) are continuous in x, the expected value is continuous in x.

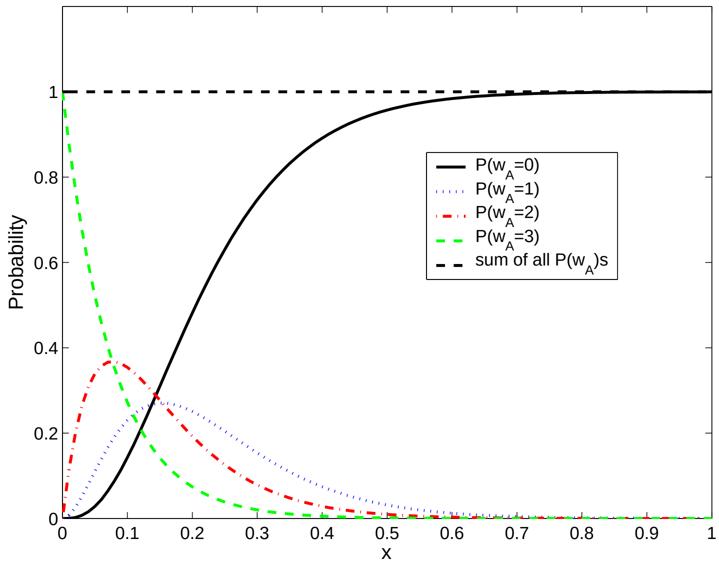

Figure 4.1 Probability of w_A warheads remaining ($w_A = 0 : 3$) at x and their sum: $\lambda_{A_T} = 3,\ \lambda_{A_{FT}} = 10,\ w = 3$

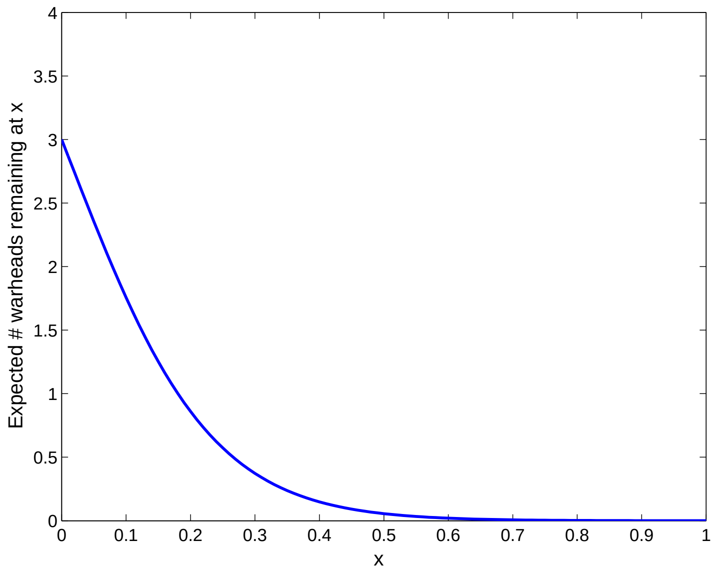

Figure 4.2 Expected number of warheads remaining at x: $\lambda_{A_T} = 3,\ \lambda_{A_{FT}} = 10,\ w = 3$

4.2.5 Probability of Additional TA's Given a Certain Number of Warheads Remaining After the Region A. Now (3.41) becomes

$$\begin{aligned}
P_t^{w_A}(x) &= \sum_{f_y=0}^{w_A-t_y-1} e^{-\lambda_{A_T}(1-x)} \frac{(\lambda_{A_T}(1-x))^{t_y}}{t_y!} e^{-\lambda_{A_{FT}}(1-x)} \frac{(\lambda_{A_{FT}}(1-x))^{f_y}}{f_y!} + \\
&\quad \frac{\lambda_{A_T}^{t_y}\lambda_{A_{FT}}^{w_A-t_y}}{(\lambda_{A_T}+\lambda_{A_{FT}})^{w_A}} \binom{w_A}{t_y} \frac{\gamma(w_A, (\lambda_{A_T}+\lambda_{A_{FT}})(1-x))}{\Gamma(w_A)} \qquad (4.86)
\end{aligned}$$

4.2.6 Probability of Mission Success. For Scenario 2, we will define mission success as having at least m TA's. We then have

$$\begin{aligned}
P_{t\geq m}^{(w)}(x) &= \int_0^x \sum_{f=0}^{w-m} e^{-(\lambda_{A_T}+\lambda_{A_{FT}})z} \frac{(\lambda_{A_T}z)^{m-1}}{(m-1)!} \frac{(\lambda_{A_{FT}}z)^f}{f!} \lambda_{A_T} dz \qquad (4.87) \\
&= \sum_{f=0}^{w-m} \binom{m+f-1}{f} \frac{\lambda_{A_T}^m \lambda_{A_{FT}}^f}{(\lambda_{A_T}+\lambda_{A_{FT}})^{m+f}} \frac{\gamma(m+f, (\lambda_{A_T}+\lambda_{A_{FT}})x)}{\Gamma(m+f)} \ . \qquad (4.88)
\end{aligned}$$

$$\begin{aligned}
E[x_s] &= \int_0^x \sum_{f=0}^{w-m} z e^{-(\lambda_{A_T}+\lambda_{A_{FT}})z} \frac{(\lambda_{A_T}z)^{m-1}}{(m-1)!} \frac{(\lambda_{A_{FT}}z)^f}{f!} \lambda_{A_T} dz \qquad (4.89) \\
&= \sum_{f=0}^{w-m} \binom{m+f}{f} m \frac{\lambda_{A_T}^m \lambda_{A_{FT}}^f}{(\lambda_{A_T}+\lambda_{A_{FT}})^{m+f+1}} \frac{\gamma(m+f+1, (\lambda_{A_T}+\lambda_{A_{FT}}))}{\Gamma(m+f+1)} \ . \qquad (4.90)
\end{aligned}$$

4.2.7 Probability of Mission Failure. Plugging (4.72) and (4.73) into (3.51) we calculate

$$\begin{aligned}
P_{t<m}^{(w)}(A_s) &= \sum_{t=0}^{m-1} \lambda_{A_T}^t \lambda_{A_{FT}}^{w-t} \frac{w}{t!(w-t)!} \int_0^1 \left\{ e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x} x^{w-1} \right\} dx + \\
&\quad \sum_{t=0}^{m-1} \sum_{f=0}^{w-t-1} e^{-(\lambda_{A_T}+\lambda_{A_{FT}})} \lambda_{A_T}^t \lambda_{A_{FT}}^f \frac{1}{t!f!} \qquad (4.91) \\
&= \sum_{t=0}^{m-1} \lambda_{A_T}^t \lambda_{A_{FT}}^{w-t} \binom{w}{t} \frac{\gamma(w, (\lambda_{A_T}+\lambda_{A_{FT}}))}{\Gamma(w)(\lambda_{A_T}+\lambda_{A_{FT}})^w} + \\
&\quad \sum_{t=0}^{m-1} \sum_{f=0}^{w-t-1} e^{-(\lambda_{A_T}+\lambda_{A_{FT}})} \lambda_{A_T}^t \lambda_{A_{FT}}^f \frac{1}{t!f!} \ . \qquad (4.92)
\end{aligned}$$

4.2.8 Expected Vehicle Longevity.

$$\begin{aligned} E[x_{vl}] &= \lim_{n \to +\infty} \sum_{i=1}^{n} z_i \sum_{t=0}^{w-1} \left[e^{-(\lambda_{A_T}+\lambda_{A_{FT}})z_i} \frac{(\lambda_{A_T} z_i)^t}{t!} \frac{(\lambda_{A_{FT}} z_i)^{w-1-t}}{(w-1-t)!} \lambda_{A_T} \Delta_i z + \right. \\ & \left. e^{-(\lambda_{A_T}+\lambda_{A_{FT}})z_i} \frac{(\lambda_{A_T} z_i)^t}{t!} \frac{(\lambda_{A_{FT}} z_i)^{w-1-t}}{(w-1-t)!} \lambda_{A_{FT}} \Delta_i z \right] + \\ & (1) \sum_{w_u=0}^{w-1} \sum_{t=0}^{w_u} \left[e^{-(\lambda_{A_T}+\lambda_{A_{FT}})} \frac{(\lambda_{A_T})^t}{t!} \frac{(\lambda_{A_{FT}})^{w_u-t}}{(w_u-t)!} \right] . \end{aligned} \tag{4.93}$$

Using the definition of the definite integral, we have

$$\begin{aligned} E[x_{vl}] &= \int_0^1 z \sum_{t=0}^{w-1} \left[e^{-(\lambda_{A_T}+\lambda_{A_{FT}})z} \frac{(\lambda_{A_T} z)^t}{t!} \frac{(\lambda_{A_{FT}} z)^{w-1-t}}{(w-1-t)!} \lambda_{A_T} dz + \right. \\ & \left. e^{-(\lambda_{A_T}+\lambda_{A_{FT}})z} \frac{(\lambda_{A_T} z)^t}{t!} \frac{(\lambda_{A_{FT}} z)^{w-1-t}}{(w-1-t)!} \lambda_{A_{FT}} dz \right] + \\ & (1) \sum_{w_u=0}^{w-1} \sum_{t=0}^{w_u} \left[e^{-(\lambda_{A_T}+\lambda_{A_{FT}})} \frac{(\lambda_{A_T})^t}{t!} \frac{(\lambda_{A_{FT}})^{w_u-t}}{(w_u-t)!} \right] \end{aligned} \tag{4.94}$$

$$\begin{aligned} &= \frac{w}{(\lambda_{A_T}+\lambda_{A_{FT}})} \frac{\gamma(w+1, (\lambda_{A_T}+\lambda_{A_{FT}}))}{\Gamma(w+1)} + \\ & \sum_{w_u=0}^{w-1} \sum_{t=0}^{w_u} e^{-(\lambda_{A_T}+\lambda_{A_{FT}})} \frac{(\lambda_{A_T})^t}{t!} \frac{(\lambda_{A_{FT}})^{w_u-t}}{(w_u-t)!} . \end{aligned} \tag{4.95}$$

4.3 Scenario 3

In this scenario we have N targets uniformly distributed amongst a Poisson field of false targets. As stated earlier, we will only look at the probability of an exact number of target attacks and false target attacks.

4.3.1 Probability of an Exact Number of Target and False Target Attacks.

As in the other scenarios, we must again split look at this probability for two cases.

Case 1 ($t+f=w$): Recall that we have to first determine the terms in (3.14). This equation is repeated here:

$$p_{t,f}^{(t+f=w)}(x)dx = P\left(\mathcal{T}_{t-1,A} \cap \mathcal{T}_{1,\Delta A}\right) P\left(\mathcal{F}_{f,A}\right) + P\left(\mathcal{T}_{t,A}\right) P\left(\mathcal{F}_{f-1,A} \cap \mathcal{F}_{1,\Delta A}\right) .$$

Since we have a finite number of targets, the TA events in $(\mathcal{T}_{t-1,A} \cap \mathcal{T}_{1,\Delta A})$ are not independent. We can determine this joint probability using conditional probabilities;

$$P\left(\mathcal{T}_{t-1,A} \cap \mathcal{T}_{1,\Delta A}\right) = P\left(\mathcal{T}_{t-1,A} | \mathcal{T}_{1,\Delta A}\right) P\left(\mathcal{T}_{1,\Delta A}\right) . \tag{4.96}$$

To have $t-1$ TA in A means we have at least $t-1$ T in A. Also, since we have only N T's in all of $\mathsf{A_s}$ then to have a TA in $\Delta\mathsf{A}$ means we can have at most $N-1$ T in A. In addition, recall that there can occur at most one event in a uniform distribution (see Appendix C). Therefore,

$$P\left(\mathcal{T}_{t-1,A} | \mathcal{T}_{1,\Delta A}\right) = P\left(\bigcup_{\bar{t}=t-1}^{N-1} \left\{T_{\bar{t},A} \cap RT_{t-1}\right\}\right) , \tag{4.97}$$

$$P\left(\mathcal{T}_{1,\Delta A}\right) = P_{TR} N dx . \tag{4.98}$$

Therefore,

$$P\left(\mathcal{T}_{t-1,A} \cap \mathcal{T}_{1,\Delta A}\right) = \sum_{\bar{t}=t-1}^{N-1} \left[\binom{N-1}{\bar{t}} (x)^{\bar{t}} (1-x)^{N-1-\bar{t}} \binom{\bar{t}}{t-1} \times (P_{TR})^{t-1} (1-P_{TR})^{\bar{t}-(t-1)}\right] P_{TR} N dx . \tag{4.99}$$

Similarly

$$P\left(\mathcal{T}_{t,A}\right) = \sum_{\bar{t}=t}^{N} \binom{N}{\bar{t}} (x)^{\bar{t}} (1-x)^{N-\bar{t}} \binom{\bar{t}}{t} (P_{TR})^{t} (1-P_{TR})^{\bar{t}-t} . \tag{4.100}$$

Since the FT's are distributed according to a Poisson distribution, we have

$$P\left(\mathcal{F}_{f-1,A} \cap \mathcal{F}_{1,\Delta A}\right) = e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^{f-1}}{(f-1)!} \lambda_{A_{FT}} dx, \tag{4.101}$$

$$P\left(\mathcal{F}_{f,A}\right) = e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^{f}}{f!} . \tag{4.102}$$

Therefore,

$$
\begin{aligned}
&\int_0^x p_{t,f}^{(t+f=w)}(x)dx \\
= \; & \int_0^x \sum_{\bar{t}=t-1}^{N-1} \left[\binom{N-1}{\bar{t}} (x)^{\bar{t}} (1-x)^{N-1-\bar{t}} \binom{\bar{t}}{t-1} (P_{TR})^{t-1} (1-P_{TR})^{\bar{t}-(t-1)} \right] \\
& \times P_{TR} N dx e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} + \int_0^x \sum_{\bar{t}=t}^{N} \binom{N}{\bar{t}} (x)^{\bar{t}} (1-x)^{N-\bar{t}} \binom{\bar{t}}{t} \\
& \times (P_{TR})^t (1-P_{TR})^{\bar{t}-t} e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^{f-1}}{(f-1)!} \lambda_{A_{FT}} dx \ .
\end{aligned} \tag{4.103}
$$

Which simplifies to

$$
\begin{aligned}
&\int_0^x p_{t,f}^{(t+f=w)}(x)dx = \\
&\textstyle\int_0^x (t+f) \sum_{\bar{t}=t}^{N} \left[\binom{N}{\bar{t}} (x)^{\bar{t}-1} (1-x)^{N-\bar{t}} \binom{\bar{t}}{t} (P_{TR})^t (1-P_{TR})^{\bar{t}-t} \right] e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} dx \ , \\
&\textstyle = (t+f) \sum_{\bar{t}=t}^{N} \binom{N}{\bar{t}} \binom{\bar{t}}{t} (P_{TR})^t (1-P_{TR})^{\bar{t}-t} \frac{(\lambda_{A_{FT}})^f}{f!} \\
&\textstyle \times \int_0^x (x)^{\bar{t}-1} (1-x)^{N-\bar{t}} e^{-\lambda_{A_{FT}} x} x^f dx \ .
\end{aligned} \tag{4.104}
$$

The solution is

$$
\begin{aligned}
P_{t,f}^{(t+f=w)}(x) = & w \sum_{\bar{t}=t}^{N} \binom{N}{\bar{t}} \binom{\bar{t}}{t} (P_{TR})^t (1-P_{TR})^{\bar{t}-t} \frac{(\lambda_{A_{FT}})^f}{f!} \\
& \times \sum_{i=0}^{N-\bar{t}} \binom{N-\bar{t}}{i} (-1)^{N-\bar{t}-i} \frac{\gamma(f+N-i, \lambda_{A_{FT}} x)}{\lambda_{A_{FT}}^{f+N-i}}
\end{aligned} \tag{4.105}
$$

Case 2 ($t+f<w$):

$$
P_{t,f}^{(t+f<w)}(x) = \sum_{\bar{t}=t}^{N} \binom{N}{\bar{t}} (x)^{\bar{t}} (1-x)^{N-\bar{t}} \binom{\bar{t}}{t} (P_{TR})^t (1-P_{TR})^{\bar{t}-t} e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} \tag{4.106}
$$

Using the truncated binomial conversion (see Appendix D), we have

$$
P_{t,f}^{(t+f<w)}(x) = \binom{N}{t} (P_{TR} x)^t (1-P_{TR} x)^{N-t} e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} \tag{4.107}
$$

As a reminder, for Scenarios 3 through 6 we are just showing the computation of $P_{t,f}(x)$. Once this probability is computed, it is a simple matter of using various combinations of t and f to obtain other probabilities such as $P_t(x)$, P^{w_A}, and probabilities of mission success and mission failure. However, the probability of additional TA's given a certain number of warheads remaining after A cannot be obtained through combinations of t and f. This is primarily because, for this probability, we must take into account all the possible states we could be in when we are at the end of A with w_A warheads left.

4.4 Scenario 4

In this scenario we have N targets uniformly distributed amongst M false targets, also uniformly distributed. Again, we exclusively examine the probability of an exact number of target attacks and false target attacks.

4.4.1 Probability of an Exact Number of Target and False Target Attacks. As in the other scenarios, we must again split look at this probability for two cases.

Case 1 ($t + f = w$): Recall that we have to first determine the terms in (3.14). This equation is repeated here:

$$p_{t,f}^{(t+f=w)}(x)dx \quad = \quad P\left(\mathcal{T}_{t-1,A} \cap \mathcal{T}_{1,\Delta A}\right) P\left(\mathcal{F}_{f,A}\right) + P\left(\mathcal{T}_{t,A}\right) P\left(\mathcal{F}_{f-1,A} \cap \mathcal{F}_{1,\Delta A}\right) \ .$$

As in Scenario 3, we have a finite number of targets, therefore the TA events in $(\mathcal{T}_{t-1,A} \cap \mathcal{T}_{1,\Delta A})$ are not independent and evaluate the same as in Scenario 3. In Scenario 4 we do the same types of things for the FTA events in $(\mathcal{F}_{f-1,A} \cap \mathcal{F}_{1,\Delta A})$. We can determine this joint probability using conditional probabilities. Following the same procedure for the FTA's as we did for the TA's in Scenario 3, we see that

$$P\left(\mathcal{F}_{f,A}\right) = \sum_{\bar{f}=f}^{M} \binom{M}{\bar{f}} (x)^{\bar{f}} (1-x)^{M-\bar{f}} \binom{\bar{f}}{f} (1-P_{FTR})^{f} (P_{FTR})^{\bar{f}-f} \ . \tag{4.108}$$

Also

$$P\left(\mathcal{F}_{f-1,A} \cap \mathcal{F}_{1,\Delta A}\right) =$$
$$\sum_{\bar{f}=f-1}^{M-1} \left[\binom{M-1}{\bar{f}} (x)^{\bar{f}} (1-x)^{M-1-\bar{f}} \binom{\bar{f}}{f-1} (1-P_{FTR})^{f-1} (P_{FTR})^{\bar{f}-(f-1)} \right] \times$$
$$(1-P_{FTR}) M dx \ . \tag{4.109}$$

Using the truncated binomial conversion,

$$\sum_{i=c}^{N} \binom{N}{i} (A)^i (B)^{(N-i)} \binom{i}{c} (C)^c (D)^{(i-c)} = \binom{N}{c} (AC)^c \left[(AD)+B\right]^{(N-c)} \ , \tag{4.110}$$

which then makes

$$\begin{aligned} P_{t,f}^{t+f=w}(x) &= \int_0^x \Bigg(\binom{N}{t} (P_{TR}x)^t (1-P_{TR}x)^{(N-t)} \times \\ & \binom{M}{f} ((1-P_{FTR})x)^f (1-(1-P_{FTR})x)^{M-f} \frac{1}{x}(t+f) \Bigg) dx \end{aligned} \tag{4.111}$$

We then use the binomial conversion to convert the polynomials with x into a series and integrate to obtain

$$P_{t,f}^{t+f=w}(x) = \binom{N}{t} \sum_{i=0}^{N-t} \binom{N-t}{i} (-1)^{N-t-i} (P_{TR})^{N-i} \binom{M}{f} \times$$
$$\sum_{j=0}^{M-f} \binom{M-f}{j} (-1)^{M-f-j} (1-P_{FTR})^{M-j} (t+f) \frac{x^{N-i+M-j}}{N-i+M-j} \ . \tag{4.112}$$

Case 2 ($t+f<w$):

Since we can find the probability of t TA and f FTA directly,

$$P_{t,f}^{t+f<w}(x) = \sum_{\bar{t}=t}^{N} \binom{N}{\bar{t}} x^{\bar{t}} (1-x)^{N-\bar{t}} \binom{\bar{t}}{t} P_{TR}^t \left(1-P_{TR}\right)^{\bar{t}-t}$$
$$\times \sum_{\bar{f}=f}^{M} \binom{M}{\bar{f}} x^{\bar{f}} (1-x)^{M-\bar{f}} \binom{\bar{f}}{f} \left(1-P_{FTR}\right)^f \left(P_{FTR}\right)^{\bar{f}-f} \ . \tag{4.113}$$

Once again, we make use of the truncated binomial conversion to obtain

$$\begin{aligned} P_{t,f}^{t+f<w}(x) = \binom{N}{t} (P_{TR}x)^t (1 - P_{TR}x)^{N-t} \\ \times \binom{M}{f} ((1 - P_{FTR})x)^f (1 - (1 - P_{FTR})x)^{M-f} \quad . \end{aligned} \tag{4.114}$$

4.5 Scenario 5

In this scenario we have N targets distributed according to a circular normal distribution amongst a Poisson field of false targets.

If the location of an object is distributed according to circular normal distribution centered about the origin with a variance σ_T^2, then the probability of the object being at the point (x, y) is

$$f(x, y) = \frac{1}{2\pi\sigma_T^2} e^{-\frac{x^2+y^2}{2\sigma_T^2}} \tag{4.115}$$

So to find the probability of the object being within a radius r with $r^2 = x^2 + y^2$, we convert to polar coordinates and compute as follows:

$$\begin{aligned} P(\text{object in } r) &= \int_0^r \int_0^{2\pi} \frac{1}{2\pi\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} \rho d\theta d\rho && (4.116) \\ &= \int_0^r \frac{1}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} \rho d\rho && (4.117) \\ &= 1 - e^{-\frac{r^2}{2\sigma_T^2}} && (4.118) \end{aligned}$$

So then, the probability of one object out of N objects being in an annulus with inner radius of ρ and width $d\rho$ is

$$N \frac{\rho}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} P_{TR} \tag{4.119}$$

To find the probability of an object being within a radius r of the origin, we simply integrate the annulus probability from 0 to r.

$$\begin{aligned} P(\rho) &= \int_0^r \frac{1}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} \rho d\rho & (4.120) \\ &= 1 - e^{-\frac{r^2}{2\sigma_T^2}} . & (4.121) \end{aligned}$$

From this we can see that the probability of the object being outside a circle of radius r is

$$P(\text{object outside of } r) = e^{-\frac{r^2}{2\sigma_T^2}} \tag{4.122}$$

To find the probabilities of objects following a Poisson distribution being in a circular region of radius ρ and being in an annulus whose inner diameter is ρ with a width of $d\rho$, we look at the comparable probabilities in a rectangular area.

In the rectangular area, for a Poisson distribution, we had for the area defined by x

$$P = e^{-\lambda_{FT}(1-P_{FTR})x} \frac{(\lambda_{FT}(1-P_{FTR})x)^f}{f!} \tag{4.123}$$

and for the strip defined by dx

$$P = \lambda_{FT}(1-P_{FTR})dx \ , \tag{4.124}$$

where the strip equation is the derivative of the argument in the Poisson distribution. We see that for the circular area, $\alpha_c \pi r^2 (1-P_{FTR})$ takes the place of $\lambda_{FT}(1-P_{FTR})x$. Therefore, we have

$$\begin{aligned} P\{\mathcal{F}_{f,A}\} &= e^{-\alpha_c \pi \rho^2 (1-P_{FTR})} \frac{(\alpha_c \pi \rho^2 (1-P_{FTR}))^f}{f!} \ , & (4.125) \\ P\{\mathcal{F}_{f,\Delta A}\} &= 2\alpha_c \pi \rho (1-P_{FTR}) & (4.126) \end{aligned}$$

4.5.1 Probability of an Exact Number of Target and False Target Attacks. As in the other scenarios, we must again examine this probability for two cases.

Case 1 ($t+f=w$): Recall that we have to first determine the terms in (3.14). This equation is repeated here:

$$p_{t,f}^{(t+f=w)}(x)dx \quad = \quad P\left(\mathcal{T}_{t-1,A} \cap \mathcal{T}_{1,\Delta A}\right) P\left(\mathcal{F}_{f,A}\right) + P\left(\mathcal{T}_{t,A}\right) P\left(\mathcal{F}_{f-1,A} \cap \mathcal{F}_{1,\Delta A}\right) \ .$$

Since we have a finite number of targets, the TA events in $(\mathcal{T}_{t-1,A} \cap \mathcal{T}_{1,\Delta A})$ are not independent. We can determine this joint probability using conditional probabilities as we did in Scenarios 3 and 4;

$$\begin{aligned} P(\mathcal{T}_{t-1,A_\rho}|\mathcal{T}_{1,\Delta A_\rho}) &= \sum_{i=t-1}^{N-1} \binom{N-1}{i} \left(1-e^{-\frac{\rho^2}{2\sigma_T^2}}\right)^i \left(e^{-\frac{\rho^2}{2\sigma_T^2}}\right)^{N-1-i} \\ &\quad \times \binom{i}{t-1} (P_{TR})^{t-1}(1-P_{TR})^{i-(t-1)} \ , \qquad (4.127) \\ P(\mathcal{T}_{1,\Delta A_\rho}) &= N\frac{\rho}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} P_{TR} d\rho \ . \qquad (4.128) \end{aligned}$$

Therefore,

$$\begin{aligned} P\left(\mathcal{T}_{t-1,A_\rho} \cap \mathcal{T}_{1,\Delta A_\rho}\right) = \sum_{i=t-1}^{N-1} \binom{N-1}{i} \left(1-e^{-\frac{\rho^2}{2\sigma_T^2}}\right)^i \left(e^{-\frac{\rho^2}{2\sigma_T^2}}\right)^{N-1-i} \\ \times \binom{i}{t-1} (P_{TR})^{t-1}(1-P_{TR})^{i-(t-1)} N\frac{\rho}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} P_{TR} d\rho \ . \qquad (4.129) \end{aligned}$$

So then

$$\begin{aligned} &P_{t,f}^{(t+f=w)}(A_\rho) = \\ &\int_0^r \left(\sum_{i=t-1}^{N-1} \binom{N-1}{i} \left(1-e^{-\frac{\rho^2}{2\sigma^2}}\right)^i \left(e^{-\frac{\rho^2}{2\sigma^2}}\right)^{N-1-i} \binom{i}{t-1} (P_{TR})^{t-1}(1-P_{TR})^{i-(t-1)} \times \right. \\ &\left. N\frac{\rho}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} P_{TR} e^{-\alpha_c \pi \rho^2 (1-P_{FTR})} \frac{(\alpha_c \pi \rho^2 (1-P_{FTR}))^f}{f!} \right) d\rho + \\ &\int_0^r \left(\sum_{i=t}^{N} \binom{N}{i} \left(1-e^{-\frac{\rho^2}{2\sigma^2}}\right)^i \left(e^{-\frac{\rho^2}{2\sigma^2}}\right)^{N-i} \binom{i}{t} (P_{TR})^{t}(1-P_{TR})^{i-t)} \times \right. \\ &\left. e^{-\alpha_c \pi \rho^2 (1-P_{FTR})} \frac{(\alpha_c \pi \rho^2 (1-P_{FTR}))^f - 1}{(f-1)!} 2\alpha_c \pi \rho (1-P_{FTR}) \right) d\rho \ . \qquad (4.130) \end{aligned}$$

Or put another way, define

$$\begin{aligned}
\mathbb{A}(\rho) &= \left(1-e^{-\frac{\rho^2}{2\sigma^2}}\right)P_{TR}\ , && (4.131)\\
\mathbb{A}'(\rho) &= \frac{\rho}{\sigma_T^2}e^{-\frac{\rho^2}{2\sigma_T^2}}P_{TR}\ , && (4.132)\\
\mathbb{B}(\rho) &= \alpha_c\pi\rho^2(1-P_{FTR})\ , && (4.133)\\
\mathbb{B}'(\rho) &= 2\alpha_c\pi\rho(1-P_{FTR})\ . && (4.134)
\end{aligned}$$

Then we have

$$\begin{aligned}
\dot{P}_{t,f}^{(t+f=w)}(A_\rho) &= \binom{N}{t}[\mathbb{A}(\rho)]^t[1-\mathbb{A}(\rho)]^{N-t}\\
&\quad\times e^{\mathbb{B}(\rho)}\frac{(\mathbb{B}(\rho))^f}{f!}\left(\frac{\mathbb{A}'(\rho)}{\mathbb{A}(\rho)}t+\frac{\mathbb{B}'(\rho)}{\mathbb{B}(\rho)}f\right)\ . \qquad (4.135)
\end{aligned}$$

Just for completeness, it can be shown (using binomial conversions) that

$$\begin{aligned}
&P_{t,f}^{(t+f=w)}(A_r) =\\
&\quad\frac{t(\alpha_c\pi(1-P_{FTR}))^f}{\sigma_T^2 f!}\binom{N}{t}\sum_{i=0}^{t-1}\binom{t-1}{i}\sum_{j=0}^{N-t}\binom{N-t}{j}P_{TR}^{N-j}\sum_{k=0}^{N-t-j}\binom{N-t-j}{k}\times\\
&\quad(-1)^{2N-t-i-2j-k-1}\frac{1}{2}\frac{\gamma(f+1,\left(\frac{N-i-j-k)}{2\sigma_T^2}+\alpha_c\pi(1-P_{FTR})\right)r^2)}{\left(\frac{N-i-j-k)}{2\sigma_T^2}+\alpha_c\pi(1-P_{FTR})\right)^{f+1}}\ . \qquad (4.136)
\end{aligned}$$

Note, we found that when computing the probability, it was faster to use a numerical integration routine.

Case 2 $(t+f<w)$:

Again, we can compute the probability directly,

$$\begin{aligned}
P_{t,f}^{(t+f<w)}(A_r) &= \binom{N}{t}\left(\left(1-e^{-\frac{r^2}{2\sigma^2}}\right)P_{TR}\right)^t\left(1-\left(1-e^{-\frac{r^2}{2\sigma^2}}\right)P_{TR}\right)^{N-t}\\
&\quad\times e^{-\alpha_c\pi r^2(1-P_{FTR})}\frac{(\alpha_c\pi\rho^2(1-P_{FTR}))^f}{f!}\ , && (4.137)\\
&= \binom{N}{t}(\mathbb{A}(r))^t\,(1-\mathbb{A}(r))^{N-t}\,e^{-\mathbb{B}(r)}\frac{(\mathbb{B}(r))^f}{f!}\ . && (4.138)
\end{aligned}$$

4.6 Scenario 6

In this scenario we have N targets distributed according to a circular normal distribution with variance σ_T^2 amongst a circular normal field of M false targets with variance σ_{FT}^2.

As in Scenario 5, the probability density function (pdf) for a true target is

$$f_t(x,y) = \frac{1}{2\pi\sigma_T^2} e^{-\frac{x^2+y^2}{2\sigma_T^2}} \ . \tag{4.139}$$

However, now the pdf for a false target is

$$f_f(x,y) = \frac{1}{2\pi\sigma_{FT}^2} e^{-\frac{x^2+y^2}{2\sigma_{FT}^2}} \tag{4.140}$$

4.6.1 Probability of an Exact Number of Target and False Target Attacks. As in the other scenarios, we must again split this probability into two cases.

Case 1 ($t + f = w$): Recall that we have to first determine the terms in (3.14). This equation is repeated here:

$$p_{t,f}^{(t+f=w)}(x)dx \quad = \quad P\left(\mathcal{T}_{t-1,A} \cap \mathcal{T}_{1,\Delta A}\right) P\left(\mathcal{F}_{f,A}\right) + P\left(\mathcal{T}_{t,A}\right) P\left(\mathcal{F}_{f-1,A} \cap \mathcal{F}_{1,\Delta A}\right) \ .$$

The $\mathbb{A}(\rho)$ and $\mathbb{A}'(\rho)$ are the same as for Scenario 5. We calculate the $\mathbb{B}(\rho)$ and $\mathbb{B}'(\rho)$ via similarity. That is

$$\mathrm{TA} \quad \sim \quad \mathrm{FTA} \tag{4.141}$$

$$\sigma_T \quad \sim \quad \sigma_{FT} \tag{4.142}$$

$$P_{TR} \quad \sim \quad 1 - P_{FTR} \ . \tag{4.143}$$

We then have

$$\mathbb{B}(\rho) = \left(1 - e^{-\frac{\rho^2}{2\sigma_{FT}^2}}\right)(1 - P_{FTR}) \,, \tag{4.144}$$

$$\mathbb{B}'(\rho) = \frac{\rho}{\sigma_{FT}^2} e^{-\frac{\rho^2}{2\sigma_{FT}^2}} (1 - P_{FTR}) \,. \tag{4.145}$$

The result is

$$\begin{aligned} \dot{P}_{t,f}^{(t+f=w)}(A_\rho) &= \binom{N}{t}[\mathbb{A}(\rho)]^t[1-\mathbb{A}(\rho)]^{N-t}\binom{M}{f}[\mathbb{B}(\rho)]^f[1-\mathbb{B}(\rho)]^{M-f} \\ &\quad \times \left(\frac{\mathbb{A}'(\rho)}{\mathbb{A}(\rho)}t + \frac{\mathbb{B}'(\rho)}{\mathbb{B}(\rho)}f\right) \,. \end{aligned} \tag{4.146}$$

Case 2 ($t+f$¡w): Similarly, we have

$$\begin{aligned} &P_{t,f}^{(t+f<w)}(A_\rho) = \\ &\quad \binom{N}{t}(\mathbb{A}(r))^t(1-\mathbb{A}(r))^{N-t}\binom{M}{f}(\mathbb{B}(r))^f(1-\mathbb{B}(r))^{M-f} \,. \end{aligned} \tag{4.147}$$

As in Scenario 5, this could be solved by converting the polynomials to series expressions using the binomial conversion and then using the incomplete gamma function; however, this method is computationally slower than using a standard numerical integration package.

Note that, as in Scenarios 3 and 5, Scenarios 4 and 6 are of the same form (in terms of $\mathbb{A}(\rho)$ and $\mathbb{B}(\rho)$). If one could solve (4.146) as written, both scenarios would be simultaneously solved. Then all that would be required would be to plug in the appropriate expressions for $\mathbb{A}(\rho)$ and $\mathbb{B}(\rho)$ for each scenario. However, solution of equation (4.146) is nontrivial. Instead, it was necessary to make the appropriate substitutions for $\mathbb{A}(\rho)$ and $\mathbb{B}(\rho)$ and then solve. However, as in Scenario 5, the calculation of the probability was faster when using a numerical integration routine than when using the analytical result. In this case, when using Matlab, the run time for the numerical integration was approximately 0.5 seconds, and the run time for the analytical computation was approximately 37 seconds.

V. Markov Chain Model

The probabilities of interest can be determined by modelling the system "dynamics" using Markov techniques in which the state of the system is described by an ordered pair of states ($\mathcal{T}$,$\mathcal{F}$) in which $\mathcal{T}$ is defined as the number of target attacks, and $\mathcal{F}$ is the number of false target attacks. The state probabilities of this bivariate, continuous-time Markov chain depend on normalized time (x) into the process. The instantaneous transition probabilities may depend on the time into the process, the time increment, and the previous state. They are split into two classes.

1. $P_{\mathcal{T}_t}(x, \Delta x)$: Probability of a TA in Δx given there were exactly t TA's by x.
2. $P_{\mathcal{F}_f}(x, \Delta x)$: Probability of a FTA in Δx given there were exactly f FTA's by x.

Two arbitrary points in the Markov chain are diagramed in Figure 5.1. The top portion of the Figure examines the state ($\mathcal{T} = t, \mathcal{F} = f$) where $t + f$ is less than the number of warheads the UCAV originally carried. The bottom portion of the Figure examines the state where $t + f$ is equal to the original number of warheads.

For brevity sake, we denote the state ($\mathcal{T} = t, \mathcal{F} = f$) at time x as $X_{t,f,x}$. The probability of state $X_{t,f,x}$ is then $P\{X_{t,f,x}\}$. This is the same probability we represented as $P_{t,f}(x)$ in the previous chapters. In this Markov approach, however, it is not only necessary to keep track of states and previous states; but also to clearly distinguish the state's probability from the instantaneous transition probabilities entering and exiting the

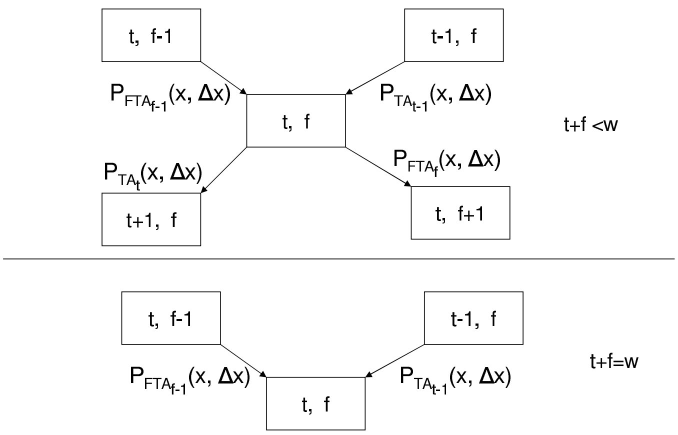

Figure 5.1 Partial Transition Rate Diagram for the Markov Chain Model

states. Note that the probability is equal to zero for any state where any of the subscripts are less than 0. In addition, when necessary, we distinguish the absorbing states from the non-absorbing states as follows:

1. $(X_{t,f,x} : t + f < w)$ is the non-absorbing state in which the total number of attacks represented by that state are less than the total number of warheads.
2. $(X_{t,f,x} : t + f = w)$ is the absorbing state in which the total number of attacks represented by that state are equal to the total number of warheads.

In Scenarios 1 and 2, we are dealing with Poisson and uniform distributions. At any time instant there can occur, at most, one event. This event can either be a TA or a FTA. We note that the last row of the Markov chain corresponds to the situation where $t + f = w$, therefore these states are absorbing states.

We will be examining the probabilities of various events. Each probability can be determined from the probability of either one state or some combination of states. If we look at the Markov chain at a given time (x), we can determine the probability of each state at that time. We can then determine the probability for each event of interest at that time.

The most elemental probability is the probability of being in a particular state. For Markov chains, the probability of being in a particular state at time x can be computed using the Chapman-Kolmogorov equation for the Markov chain.

5.1 Probability of an Exact Number of Target and False Target Attacks

Recall we have defined the state of the system as being the number of target attacks and the number of false target attacks at time x. We have also seen that we have absorbing states and non-absorbing states. The derivation of an absorbing state's probability differs from the derivation of the probability of a non-absorbing state. When $t + f < w$ (a non-absorbing state), the Chapman-Kolmogorov equation is developed as follows

$$\begin{aligned}
P\left(X_{t,f,x+\Delta x}\right) &= P\left(X_{t-1,f,x}\right) P_{\mathcal{T}_{t-1}}(x,\Delta x) + P\left(X_{t,f-1,x}\right) P_{\mathcal{F}_{f-1}}(x,\Delta x) - \\
& \quad P\left(X_{t,f,x}\right)\left(1 - P_{\mathcal{T}_t}(x,\Delta x) - P_{\mathcal{F}_f}(x,\Delta x)\right) \qquad (5.1)\\
\Rightarrow & \\
\dot{P}\left(X_{t,f,x}\right) &= \lim_{\Delta x \to 0} \frac{P\left(X_{t,f,x+\Delta x}\right) - P\left(X_{t,f,x}\right)}{\Delta x} \\
&= P\left(X_{t-1,f,x}\right) \lim_{\Delta x \to 0} \frac{P_{\mathcal{T}_{t-1}}(x,\Delta x)}{\Delta x} + P\left(X_{t,f-1,x}\right) \lim_{\Delta x \to 0} \frac{P_{\mathcal{F}_{f-1}}(x,\Delta x)}{\Delta x} - \\
& \quad P\left(X_{t,f,x}\right)\left(\lim_{\Delta x \to 0} \frac{P_{\mathcal{T}_t}(x,\Delta x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{P_{\mathcal{F}_f}(x,\Delta x)}{\Delta x}\right) . \qquad (5.2)
\end{aligned}$$

For the absorbing states ($t + f = w$), we obtain

$$\begin{aligned}
\dot{P}\left(X_{t,f,x} : t+f=w\right) &= P\left(X_{t-1,f,x}\right) \lim_{\Delta x \to 0} \frac{P_{\mathcal{T}_{t-1}}(x,\Delta x)}{\Delta x} + \\
& \quad P\left(X_{t,f-1,x}\right) \lim_{\Delta x \to 0} \frac{P_{\mathcal{F}_{f-1}}(x,\Delta x)}{\Delta x} , \qquad (5.3)
\end{aligned}$$

where $\lim_{\Delta x \to 0} \frac{P_{\mathcal{T}_{t-1}}(x,\Delta x)}{\Delta x}$ and $\lim_{\Delta x \to 0} \frac{P_{\mathcal{F}_{f-1}}(x,\Delta x)}{\Delta x}$ are the instantaneous transition probabilities.

We will now summarize these instantaneous transition probabilities for the various distributions. The development of these probabilities will be discussed in subsequent sections.

Recall we are working with a non-dimensional, normalized x. For the Poisson distribution of targets and false targets in the rectangular battle space, we have

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{P_{\mathcal{T}_t}(x,\Delta x)}{\Delta x} &= \lambda_{A_T} , \qquad (5.4)\\
\lim_{\Delta x \to 0} \frac{P_{\mathcal{F}_f}(x,\Delta x)}{\Delta x} &= \lambda_{A_{FT}} . \qquad (5.5)
\end{aligned}$$

For the uniform distribution of N targets and M false targets we have

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{P_{\mathcal{T}_t}(x,\Delta x)}{\Delta x} &= \frac{(N-t)P_{TR}}{(1-P_{TR}x)} , \qquad (5.6)\\
\lim_{\Delta x \to 0} \frac{P_{\mathcal{F}_f}(x,\Delta x)}{\Delta x} &= \frac{(M-f)(1-P_{FTR})}{(1-(1-P_{FTR})x)} . \qquad (5.7)
\end{aligned}$$

For the circular normal distribution of N targets and M false targets, we have

$$\lim_{\Delta x \to 0} \frac{P_{\mathcal{T}_t}(x, \Delta x)}{\Delta x} = \frac{(N-t)\frac{\rho}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} P_{TR}}{\left[1 - P_{TR}\left(1 - e^{-\frac{\rho^2}{2\sigma_T^2}}\right)\right]} = \frac{(N-t)\mathbb{A}'(\rho)}{(1-\mathbb{A}(\rho))} \ , \tag{5.8}$$

$$\lim_{\Delta x \to 0} \frac{P_{\mathcal{F}_f}(x, \Delta x)}{\Delta x} = \frac{(M-f)\frac{\rho}{\sigma_{FT}^2} e^{-\frac{\rho^2}{2\sigma_{FT}^2}} (1 - P_{FTR})}{\left[1 - (1 - P_{FTR})\left(1 - e^{-\frac{\rho^2}{2\sigma_{FT}^2}}\right)\right]} = \frac{(M-f)\mathbb{B}'(\rho)}{(1-\mathbb{B}(\rho))} \ , \tag{5.9}$$

where $\mathbb{A}(\rho)$, $\mathbb{A}'(\rho)$, $\mathbb{B}(\rho)$, and $\mathbb{B}'(\rho)$ are defined by (4.131), (4.132), (4.144), and (4.145).

For the Poisson distribution of false targets in a circular battle space, we have

$$\lim_{\Delta x \to 0} \frac{P_{\mathcal{F}_f}(x, \Delta x)}{\Delta x} = 2\alpha_c \pi \rho (1 - P_{FTR}) \ . \tag{5.10}$$

Therefore, for Scenario 1 we have

$$\dot{P}\left(X_{1,f,x} : t+f < w\right) = \frac{P_{TR}}{1 - P_{TR}x} P\left(X_{0,f,x}\right) + \lambda_{A_{FT}} P\left(X_{1,f-1,x}\right) - \lambda_{A_{FT}} P\left(X_{1,f,x}\right) \ , \tag{5.11}$$

$$\dot{P}\left(X_{0,f,x} : t+f < w\right) = \lambda_{A_{FT}} P\left(X_{0,f-1,x}\right) - \left(\lambda_{A_{FT}} + \frac{P_{TR}}{1 - P_{TR}x}\right) P\left(X_{0,f,x}\right) \ , \tag{5.12}$$

$$\dot{P}\left(X_{1,f,x} : t+f = w\right) = \frac{P_{TR}}{1 - P_{TR}x} P\left(X_{0,f,x} : t+f < w\right) + \lambda_{A_{FT}} P\left(X_{1,f-1,x} : t+f < w\right) \ , \tag{5.13}$$

$$\dot{P}\left(X_{0,f,x} : t+f = w\right) = \lambda_{A_{FT}} P\left(X_{0,f-1,x} : t+f < w\right) \ , \tag{5.14}$$

the solutions of which are

$$P\left(X_{1,f,x} : t+f < w\right) = P_{TR} x e^{-\lambda_{A_{FT}} x} \frac{\left(\lambda_{A_{FT}} x\right)^f}{f!} , \tag{5.15}$$

$$P\left(X_{0,f,x} : t+f < w\right) = \left(1 - P_{TR} x\right) e^{-\lambda_{A_{FT}} x} \frac{\left(\lambda_{A_{FT}} x\right)^f}{f!} , \tag{5.16}$$

$$P\left(X_{1,f,x} : t+f = w\right) = \frac{w P_{TR}}{\lambda_{A_{FT}}} \frac{\gamma\left(w, \lambda_{A_{FT}} x\right)}{\Gamma\left(w\right)} , \tag{5.17}$$

$$P\left(X_{0,f,x} : t+f = w\right) = \frac{\gamma\left(w, \lambda_{A_{FT}} x\right)}{\Gamma(w)} - P_{TR} \frac{w}{\lambda_{A_{FT}}} \frac{\gamma\left(w+1, \lambda_{A_{FT}} x\right)}{\Gamma\left(w+1\right)} , \tag{5.18}$$

where $\gamma(\alpha, z)$ is the incomplete gamma function.

For Scenario 2,

$$\begin{aligned} \dot{P}\left(X_{t,f,x} : t+f < w\right) &= \lambda_{A_T} P\left(X_{t-1,f,x}\right) + \lambda_{A_{FT}} P\left(X_{t,f-1,x}\right) - \\ &\quad \left(\lambda_{A_T} + \lambda_{A_{FT}}\right) P\left(X_{t,f,x}\right) , \end{aligned} \tag{5.19}$$

$$\begin{aligned} \dot{P}\left(X_{t,f,x} : t+f = w\right) &= \lambda_{A_T} P\left(X_{t-1,f,x} : t+f < w\right) + \\ &\quad \lambda_{A_{FT}} P\left(X_{t,f-1,x} : t+f < w\right) . \end{aligned} \tag{5.20}$$

The solution is

$$P\left(X_{t,f,x} : t+f < w\right) = \frac{\left(\lambda_{A_T} x\right)^t}{t!} \frac{\left(\lambda_{A_{FT}} x\right)^f}{f!} e^{-\left(\lambda_{A_T} + \lambda_{A_{FT}}\right) x} , \tag{5.21}$$

$$P\left(X_{t,f,x} : t+f = w\right) = \lambda_{A_T}^t \lambda_{A_{FT}}^{w-t} \binom{w}{t} \frac{\gamma\left(w, \left(\lambda_{A_T} + \lambda_{A_{FT}}\right) x\right)}{\Gamma(w) \left(\lambda_{A_T} + \lambda_{A_{FT}}\right)^w} . \tag{5.22}$$

Several probabilities can then be derived from $P\left(X_{t,f,x} : t+f < w\right)$ and $P\left(X_{t,f,x} : t+f = w\right)$. The first probability we will derive is the probability of a specific number of target attacks in normalized time x. This probability (while perhaps important in its own right) is most useful in the calculation of other probabilities. Specifically, an operational commander may wish to weigh the cost versus benefit of starting a particular search and attack operation. To assist the commander in making this decision, several probabilities could be used as determining factors. In particular, the probability of at least a specified number of attacks or the expected number of target attacks could be used by the commander when determining the value of starting a particular operation. Each of these first requires the calculation of the probability of a specified number of target attacks.

5.2 Probability of Specified Number of Target Attacks

The probability of a specified number of target attacks is determined by summing the probabilities of all the states which have that number of attacks. This requires a summation over the number of false target attacks:

$$\begin{aligned} P\left(X_{t,f\geq 0,x}\right) &= \sum_{f=0}^{w-t} P\left(X_{t,f,x}\right) \ , \\ &= \sum_{f=0}^{w-t-1} P\left(X_{t,f,x} : t+f < w\right) + P\left(X_{t,w-t,x} : t+f = w\right) \ , \end{aligned} \tag{5.23}$$

where we adopt the convention that if the upper limit on the summation is less than the lower limit, the summation is equal to zero. We also adopt a convention that the notation $f \geq 0$ indicates that we include all allowable f's greater than or equal to 0; i.e. $\bigcup_{f=0}^{w-t} f$. For Scenario 1 this becomes

$$P\left(X_{1,f\geq 0,x}\right) = \sum_{f=0}^{w-2} \left\{ P_{TR} x e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} \right\} + \frac{w P_{TR}}{\lambda_{A_{FT}}} \frac{\gamma(w, \lambda_{A_{FT}} x)}{\Gamma(w)} \ , \tag{5.24}$$

$$\begin{aligned} P\left(X_{0,f\geq 0,x}\right) &= \sum_{f=0}^{w-1} \left\{ \left(1 - P_{TR} x\right) e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} \right\} + \frac{\gamma\left(w, \lambda_{A_{FT}} x\right)}{\Gamma(w)} - \\ & P_{TR} \frac{w}{\lambda_{A_{FT}}} \frac{\gamma\left(w+1, \lambda_{A_{FT}} x\right)}{\Gamma\left(w+1\right)} \ . \end{aligned} \tag{5.25}$$

For Scenario 2, we have

$$\begin{aligned} P\left(X_{t,f\geq 0,x}\right) &= \sum_{f=0}^{w-t-1} \left\{ \frac{(\lambda_{A_T} x)^t}{t!} \frac{(\lambda_{A_{FT}} x)^f}{f!} e^{-(\lambda_{A_T} + \lambda_{A_{FT}})x} \right\} + \\ & \lambda_{A_T}^t \lambda_{A_{FT}}^{w-t} \binom{w}{t} \frac{1}{\left(\lambda_{A_T} + \lambda_{A_{FT}}\right)^w} \frac{\gamma(w, \left(\lambda_{A_T} + \lambda_{A_{FT}}\right) x)}{\Gamma(w)} \ . \end{aligned} \tag{5.26}$$

The expected number of target attacks is then

$$E[t] = \sum_{t=0}^{w} tP\left(X_{t,f\geq 0,x}\right) \ . \tag{5.27}$$

Now we can compute the probability of at least a specified number of target attacks.

5.3 Probability of at Least a Specified Number of Target Attacks

The probability of at least ξ TA in A is the summation:

$$P\left(X_{t\geq\xi,f\geq 0,x}\right) \ = \ \sum_{t=\xi}^{w} P\left(X_{t,f\geq 0,x}\right) \ . \tag{5.28}$$

For Scenario 1, the probability is inconsequential since there is only one target. For Scenario 2, however, this becomes

$$\begin{aligned} P\left(X_{t\geq\xi,f\geq 0,x}\right) \ &= \ \sum_{t=\xi}^{w-1}\sum_{f=0}^{w-t-1}\frac{(\lambda_{A_T}x)^t}{t!}\frac{(\lambda_{A_{FT}}x)^f}{f!}e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x} + \\ &\quad \sum_{t=\xi}^{w}\lambda_{A_T}^{t}\lambda_{A_{FT}}^{w-t}\binom{w}{t}\frac{\gamma(w,(\lambda_{A_T}+\lambda_{A_{FT}})\,x)}{\Gamma(w)\,(\lambda_{A_T}+\lambda_{A_{FT}})^{w}} \ , \end{aligned} \tag{5.29}$$

or equivalently

$$\begin{aligned} P\left(X_{t\geq\xi,f\geq 0,x}\right) \ &= \ 1-\sum_{t=0}^{\xi-1}\sum_{f=0}^{w-t-1}\frac{(\lambda_{A_T}x)^t}{t!}\frac{(\lambda_{A_{FT}}x)^f}{f!}e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x} - \\ &\quad \sum_{t=0}^{\xi-1}\lambda_{A_T}^{t}\lambda_{A_{FT}}^{w-t}\binom{w}{t}\frac{\gamma(w,(\lambda_{A_T}+\lambda_{A_{FT}})\,x)}{\Gamma(w)\,(\lambda_{A_T}+\lambda_{A_{FT}})^{w}} \ . \end{aligned} \tag{5.30}$$

In addition, of potential interest to a tactical commander is knowing the expected number of warheads left after searching a percentage of the battle space. This information could be useful in planning area coverage using multiple UCAVs.

5.4 *Probability of a Certain Number of Warheads Remaining After Region* A *Has Been Searched*

We designate $\mathcal{W}_{\mathcal{A}}$ as the random variable representing the remaining number of warheads after covering A. A realization of the random variable is w_A. The probability of w_A warheads left after A (denoted P^{w_A}) is equivalently stated as the probability of $w - w_A$ attacks (whether TA or FTA) in A:

$$P^{w_A}(x) \equiv P\left(\mathcal{W}_{\mathcal{A}} = w_A\right) \tag{5.31}$$

$$= P\left(w - w_A \text{ attacks after } x\right) \ . \tag{5.32}$$

By assuming $w_A \neq 0$ we obtain the following for Scenario 1:

$$\begin{aligned} P^{w_A}(x) &= \sum_{t=0}^{1} P\left(X_{t,f=w-w_A-t,x} : t+f < w\right) \ , \\ &= P\left(X_{0,f=w-w_A,x} : t+f < w\right) + P\left(X_{1,f=w-w_A-1,x} : t+f < w\right) \ , \\ &= \left(1 - P_{TR}x\right) e^{-\lambda_{A_{FT}}x} \frac{\left(\lambda_{A_{FT}}x\right)^{w-w_A}}{(w-w_A)!} + \\ &\quad P_{TR}xe^{-\lambda_{A_{FT}}x} \frac{\left(\lambda_{A_{FT}}x\right)^{w-w_A-1}}{(w-w_A-1)!} \ , \end{aligned} \tag{5.33}$$

If $w_A = 0$ then

$$\begin{aligned} P^{w_A=0}(x) &= \sum_{t=0}^{1} P\left(X_{t,f=w-t,x} : t+f = w\right) \ , \\ &= P\left(X_{0,w-0,x} : t+f = w\right) + P\left(X_{1,w-1,x} : t+f = w\right) \ , \\ &= \frac{\gamma(w, \lambda_{A_{FT}}x)}{\Gamma(w)} - P_{TR}\frac{w}{\lambda_{A_{FT}}}\frac{\gamma(w+1, \lambda_{A_{FT}}x)}{\Gamma(w+1)} + \\ &\quad \frac{wP_{TR}}{\lambda_{A_{FT}}}\frac{\gamma(w, \lambda_{A_{FT}}x)}{\Gamma(w)} \ . \end{aligned} \tag{5.34}$$

For Scenario 2 we have the following for $w_A \neq 0$ and $w_A = 0$:

$$P^{w_A>0}(x) = \sum_{t=0}^{w-w_A} \frac{(\lambda_{A_T}x)^t}{t!} \frac{(\lambda_{A_{FT}}x)^{w-w_A-t}}{(w-w_A-t)!} e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x} \tag{5.35}$$

$$= e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x} \frac{((\lambda_{A_T}+\lambda_{A_{FT}})x)^{w-w_A}}{(w-w_A)!} \tag{5.36}$$

$$P^{w_A=0}(x) = \sum_{t=0}^{w} \lambda_{A_T}^t \lambda_{A_{FT}}^{w-t} \binom{w}{t} \frac{1}{(\lambda_{A_T}+\lambda_{A_{FT}})^w} \frac{\gamma(w,(\lambda_{A_T}+\lambda_{A_{FT}})\,x)}{\Gamma(w)} \tag{5.37}$$

$$= \frac{\gamma(w,(\lambda_{A_T}+\lambda_{A_{FT}})x)}{\Gamma(w)} . \tag{5.38}$$

Once we have covered region A and realize we have w_A warheads left, we can then look at how many TA's we can expect in the remaining region. This remaining region we denote as $\mathsf{A_m}$, where $\mathsf{A_m} \equiv \mathsf{A_s} - \mathsf{A}$. Its area is $A_m = A_s - A$. As x marked our passage through $\mathsf{A_s}$, so y will mark our passage through $\mathsf{A_m}$. That is, y is the proportion of $\mathsf{A_m}$ we have already covered when looking at the Markov chain. At the end of A, $y = 0$ and y increases to $y = 1$ at the end of $\mathsf{A_s}$. Also, we will denote the number of TA's in $\mathsf{A_m}$ as t_y.

5.5 Probability of Additional TA's Given $\mathcal{W}_A$ Warheads Remaining After A

At the end of A, any state such that $t + f = w - w_A$ leaves us w_A warheads. Each of these states is a new starting point when looking at the possible number of target attacks in $\mathsf{A_m}$. In essence, we have new Markov chains, one Markov chain for each possible state that gives us w_A warheads after A. As with the previous Markov chains, the instantaneous transition probabilities may depend on the previous state. In addition, the transition rate may also depend on the state we were in at the end of A, i.e. the probability of a state in $\mathsf{A_m}$ will be conditioned on a previous state as well as the state which started the new Markov chain. To allow for this, we replace our old instantaneous transition probability of $P_{\mathcal{T}_{t-1}}(x, \Delta x)$ with $P\left(\mathcal{T}_{1,\Delta y} | X_{t_y-1,f_y,y} \cap X_{t,f,x}\right)$ which is the probability of a TA in Δy given $t_y - 1$ TA's in y and being in state $X_{t,f}$ at end of A (i.e. $X_{t,f,x}$). Similarly, we also now have $P\left(\mathcal{F}_{1,\Delta y} | X_{t_y,f_y-1,y} \cap X_{t,f,x}\right)$.

Using these conditional probabilities in the Chapman-Kolmogorov equation yields

$$\begin{aligned}
\dot{P}\left(X_{t_y,f_y,y}|X_{t,f,x}\right) &= \lim_{\Delta y\to 0}\frac{P\left(X_{t_y,f_y,(y+\Delta y)}|X_{t,f,x}\right)-P\left(X_{t_y,f_y,y}|X_{t,f,x}\right)}{\Delta y}\ , \\
&= P\left(X_{t_y-1,f_y,y}|X_{t,f,x}\right)\lim_{\Delta y\to 0}\frac{P\left(\mathcal{T}_{1,\Delta y}|X_{t_y-1,f_y,y}\cap X_{t,f,x}\right)}{\Delta y}+ \\
&\quad P\left(X_{t_y,f_y-1,y}|X_{t,f,x}\right)\lim_{\Delta y\to 0}\frac{P\left(\mathcal{F}_{1,\Delta y}|X_{t_y,f_y-1,y}\cap X_{t,f,x}\right)}{\Delta y}- \\
&\quad P\left(X_{t_y,f_y,y}|X_{t,f,x}\right)\begin{pmatrix}\lim_{\Delta y\to 0}\frac{P\left(\mathcal{T}_{1,\Delta y}|X_{t_y,f_y,y}\cap X_{t,f,x}\right)}{\Delta y}+ \\ \lim_{\Delta y\to 0}\frac{P\left(\mathcal{F}_{1,\Delta y}|X_{t_y,f_y,y}\cap X_{t,f,x}\right)}{\Delta y}\end{pmatrix}
\end{aligned} \tag{5.39}$$

Equation (5.39) applies to the situation where $t_y + f_y < w_A$. To make this equation applicable to $t_y + f_y = w_A$, simply add back in the last term (there is no outflow from a state such that $t_y + f_y = w_A$).

To evaluate (5.39), we need to examine the terms of the form

$$\lim_{\Delta y\to 0}\frac{P\left(\mathcal{T}_{1,\Delta y}|X_{t_y,f_y,y}\cap X_{t,f,x}\right)}{\Delta y}\ .$$

In evaluating this expression (and a similar expression for FTA), we take advantage of the fact that the TA's and FTA's are independent of each other assuming there are enough warheads (similarly, one TA is independent of another TA and the same for the FTA's):

$$\begin{aligned}
\lim_{\Delta y\to 0}\frac{P\left(\mathcal{T}_{1,\Delta y}|X_{t_y,f_y,y}\cap X_{t,f,x}\right)}{\Delta y} &= \lim_{\Delta y\to 0}\frac{P\left(\mathcal{T}_{1,\Delta y}|\mathcal{T}_{t_y,y}\cap\mathcal{F}_{f_y,y}\cap\mathcal{T}_{t,x}\cap\mathcal{F}_{f,x}\right)}{\Delta y}\ , \\
&= \lim_{\Delta y\to 0}\frac{P\left(\mathcal{T}_{1,\Delta y}|\mathcal{T}_{t_y,y}\cap\mathcal{T}_{t,x}\right)}{\Delta y}\ , \\
&= \frac{P\left(\mathcal{T}_{t_y,y}\cap\mathcal{T}_{t,x}|\mathcal{T}_{1,\Delta y}\right)}{P\left(\mathcal{T}_{t_y,y}\cap\mathcal{T}_{t,x}\right)}\lim_{\Delta y\to 0}\frac{P\left(\mathcal{T}_{1,\Delta y}\right)}{\Delta y}\ .
\end{aligned} \tag{5.40}$$

Similarly,

$$\lim_{\Delta y\to 0}\frac{P\left(\mathcal{F}_{1,\Delta y}|X_{t_y,f_y,y}\cap X_{t,f,x}\right)}{\Delta y} = \frac{P\left(\mathcal{F}_{f_y,y}\cap\mathcal{F}_{f,x}|\mathcal{F}_{1,\Delta y}\right)}{P\left(\mathcal{F}_{f_y,y}\cap\mathcal{F}_{f,x}\right)}\lim_{\Delta y\to 0}\frac{P\left(\mathcal{F}_{1,\Delta y}\right)}{\Delta y}\ . \tag{5.41}$$

Also recall that $t + f = w - w_A$. Assuming that with these last two equations we can solve (5.39), we would then have $P\left(X_{t_y,f_y,y}|X_{t,w-w_A-t,x}\right)$.

If we knew our state when we left A we could then make the following calculation;

$$P\left(\mathcal{T}_{t_y}|X_{t,w-w_A-t,x}\right) = \sum_{f_y=0}^{w_A-t_y} P\left(X_{t_y,f_y,y}|X_{t,w-w_A-t,x}\right) \ . \tag{5.42}$$

However, when all we know is the number of warheads left after A we must include all the possible states we could have been in after A. To do so, we shall make use of the concept of the conditional probability as well as the fact that

$$P\left(\mathcal{T}_{t_y} \cap X_{t,w-w_A-t,x}\right) = \sum_{f_y=0}^{w_A-t_y} P\left(X_{t_y,f_y,y}|X_{t,w-w_A-t,x}\right) P\left(X_{t,w-w_A-t,x}\right) \ . \tag{5.43}$$

We shall use the notation $P\left(\mathcal{T}_{t_y}|\mathcal{W}_{\mathcal{A}} = w_A\right)$ to denote $P\left(\mathcal{T}_{t_y}|\bigcup_{t=0}^{w-w_A} X_{t,w-w_A-t,x}\right)$. That is, the probability of t_y TA's assuming w_A warheads left after x (or $w - w_A$ attacks in x):

$$P\left(\mathcal{T}_{t_y}|\mathcal{W}_{\mathcal{A}} = w_A\right) = \frac{\sum\limits_{t=0}^{w-w_A}\sum\limits_{f_y=0}^{w_A-t_y} P\left(X_{t_y,f_y,y}|X_{t,w-w_A-t,x}\right) P\left(X_{t,w-w_A-t,x}\right)}{\sum\limits_{t=0}^{w-w_A} P\left(X_{t,w-w_A-t,x}\right)} \ , \tag{5.44}$$

$$P\left(\mathcal{T}_{t_y\geq\xi}|\mathcal{W}_{\mathcal{A}} = w_A\right) = \frac{\sum\limits_{t_y=\xi}^{w_A}\sum\limits_{t=0}^{w-w_A}\sum\limits_{f_y=0}^{w_A-t_y} P\left(X_{t_y,f_y,y}|X_{t,w-w_A-t,x}\right) P\left(X_{t,w-w_A-t,x}\right)}{\sum\limits_{t=0}^{w-w_A} P\left(X_{t,w-w_A-t,x}\right)} \tag{5.45}$$

5.5.1 Scenario 1. For Scenario 1, to have a TA in $\mathsf{A_m}$ means we could not have a TA by x (nor could we have a TA by the end of y). Therefore, $t = 0$ (and $t_y = 0$) in

(5.40), and (5.41). For Scenario 1, we also have the following;

$$
\begin{aligned}
P\left(\mathcal{T}_{1,\Delta y}\right) &= P_{TR}\Delta y \ , && (5.46)\\
P\left(\mathcal{T}_{0,y}\cap\mathcal{T}_{0,x}|\mathcal{T}_{1,\Delta y}\right) &= 1 \ , && (5.47)\\
P\left(\mathcal{T}_{1,y}\cap\mathcal{T}_{0,x}|\mathcal{T}_{1,\Delta y}\right) &= 0 \ , && (5.48)\\
P\left(\mathcal{T}_{0,y}\cap\mathcal{T}_{0,x}\right) &= 1-P_{TR}x-P_{TR}y \ . && (5.49)
\end{aligned}
$$

And since the FTA's follow a Poisson distribution, the number of FTA's in any area are independent of the number of FTA's in any other area. Therefore,

$$
\lim_{\Delta y\to 0}\frac{P\left(\mathcal{F}_{1,\Delta y}|X_{t_y,f_y,y}\cap X_{t,f,x}\right)}{\Delta y} = \lambda_{A_{FT}} \ . \qquad (5.50)
$$

In the Markov chain that we are examining in $\mathsf{A_m}$, the probability equals zero for any state which has a subscript that is less than zero. With that in mind, we note that the Markov chain in $\mathsf{A_m}$ is composed of two types of states. One where there are no TA's in $\mathsf{A_m}$, the other where there is one TA in $\mathsf{A_m}$.

Utilizing (5.46) thru (5.50), the $t_y = 0$ situation has the following Chapman-Kolmogorov equation and solution;

$$
\begin{aligned}
\dot{P}\left(X_{0,f_y,y}|X_{0,f,x}: t_y+f_y<w_A\right) &= P\left(X_{0,f_y-1,y}|X_{0,f,x}\right)\lambda_{A_{FT}} - \\
&\quad P\left(X_{0,f_y,y}|X_{0,f,x}\right)\left(\frac{P_{TR}}{1-P_{TR}x-P_{TR}y}+\right.\\
&\quad \left.\lambda_{A_{FT}}\right) \ , && (5.51)\\
P\left(X_{0,f_y,y}|X_{0,f,x}: t_y+f_y<w_A\right) &= \frac{1-P_{TR}x-P_{TR}y}{1-P_{TR}x}e^{-\lambda_{A_{FT}}y}\frac{(\lambda_{A_{FT}}y)^{f_y}}{f_y!} \ . && (5.52)
\end{aligned}
$$

Now when $t_y = 1$ we have the following;

$$
\begin{aligned}
\dot{P}\left(X_{1,f_y,y}|X_{0,f,x}: t_y+f_y<w_A\right) &= P\left(X_{0,f_y,y}|X_{0,f,x}\right)\frac{P_{TR}}{1-P_{TR}x-P_{TR}y} + \\
&\quad P\left(X_{1,f_y-1,y}|X_{0,f,x}\right)\lambda_{A_{FT}} - \\
&\quad P\left(X_{1,f_y,y}|X_{0,f,x}\right)\lambda_{A_{FT}} \ , && (5.53)\\
P\left(X_{1,f_y,y}|X_{0,f,x}: t_y+f_y<w_A\right) &= \frac{P_{TR}}{1-P_{TR}x}ye^{-\lambda_{A_{FT}}y}\frac{(\lambda_{A_{FT}}y)^{f_y}}{f_y!} \ . && (5.54)
\end{aligned}
$$

To solve (5.39) when $t_y + f_y = w_A$ in closed form, we will make use of the incomplete gamma function. The solutions are then

$$P\left(X_{0,f_y,y} | X_{0,f,x} : t_y + f_y = w_A\right) = \frac{\gamma(w_A, \lambda_{A_{FT}} y)}{\Gamma(w_A)} - \frac{P_{TR}}{(1 - P_{TR}x)} \frac{w_A}{\lambda_{A_{FT}}} \frac{\gamma(w_A + 1, \lambda_{A_{FT}} y)}{\Gamma(w_A + 1)} , \quad (5.55)$$

and

$$P\left(X_{1,f_y,y} | X_{0,f,x} : t_y + f_y = w_A\right) = \frac{w_A P_{TR}}{(1 - P_{TR}x)\,\lambda_{A_{FT}}} \frac{\gamma(w_A, \lambda_{A_{FT}} y)}{\Gamma(w_A)} . \quad (5.56)$$

Equations (5.51) through (5.56) provides the probabilities for the states in the new Markov chain which start from a $\mathcal{T} = 0$ state. We need to compute the same probabilities for the Markov chain which starts from a $\mathcal{T} = 1$ state.

Since Scenario 1 has only 1 target, it is easy to see that

$$P\left(X_{1,f_y,y} | X_{1,f,x} : t_y + f_y < w_A\right) = 0 , \quad (5.57)$$

$$P\left(X_{1,f_y,y} | X_{1,f,x} : t_y + f_y = w_A\right) = 0 . \quad (5.58)$$

When $t_y = 0$ we have the following:

$$\dot{P}\left(X_{0,f_y,y} | X_{1,f,x} : t_y + f_y < w_A\right) = P\left(X_{0,f_y-1,y} | X_{1,f,x}\right) \lambda_{A_{FT}} - P\left(X_{0,f_y,y} | X_{1,f,x}\right) \lambda_{A_{FT}} , \quad (5.59)$$

$$P\left(X_{0,f_y,y} | X_{1,f,x} : t_y + f_y < w_A\right) = e^{-\lambda_{A_{FT}} y} \frac{(\lambda_{A_{FT}} y)^{f_y}}{f_y!} . \quad (5.60)$$

When $t_y + f_y = w_A$, we have

$$P\left(X_{0,f_y,y} | X_{1,f,x} : t_y + f_y = w_A\right) = \frac{\gamma(w_A, y)}{\Gamma(w_A)} \quad (5.61)$$

Substituting the appropriate expressions into (5.44) and simplifying, we have

$$\begin{aligned}
P\left(\mathcal{T}_{t_y=0}|\mathcal{W}_{\mathcal{A}}=w_A\right) &= \\
&\sum_{f_y=0}^{w_A-1}\left(1-\frac{P_{TR}\lambda_{A_{FT}}y}{\left(1-P_{TR}x\right)\lambda_{A_{FT}}+P_{TR}\left(w-w_A\right)}\right)e^{-\lambda_{A_{FT}}y}\frac{\left(\lambda_{A_{FT}}y\right)^{f_y}}{f_y!}+ \\
&-\frac{P_{TR}w_A}{\left(1-P_{TR}x\right)\lambda_{A_{FT}}+P_{TR}\left(w-w_A\right)}\frac{\gamma(w_A+1,\lambda_{A_{FT}}y)}{\Gamma(w_A+1)}+ \\
&\frac{\gamma(w_A,\lambda_{A_{FT}}y)}{\Gamma(w_A)}\ , \qquad (5.62) \\
P\left(\mathcal{T}_{t_y=1}|\mathcal{W}_{\mathcal{A}}=w_A\right) &= \frac{P_{TR}}{\left(1-P_{TR}x\right)\lambda_{A_{FT}}+P_{TR}\left(w-w_A\right)}\times \\
&\left[\sum_{f_y=0}^{w_A-2}e^{-\lambda_{A_{FT}}y}\frac{\left(\lambda_{A_{FT}}y\right)^{f_y+1}}{f_y!}+w_A\frac{\gamma(w_A,\lambda_{A_{FT}}y)}{\Gamma(w_A)}\right]\ . \qquad (5.63)
\end{aligned}$$

We see that (5.62) and (5.63) is the same as (4.49) and (4.48) when we set $y=1-x$.

5.5.2 Scenario 2. As stated in Scenario 1, when dealing with Poisson distribution of FTA's (and now TA's), the number of attacks in an area are independent of the number of attacks in any other area. Therefore, (5.39) and its solution become

$$\begin{aligned}
&\dot{P}\left(X_{t_y,f_y,y}|X_{t,f,x}:t_y+f_y<w_A\right)= \\
&\quad P\left(X_{t_y-1,f_y,y}\right)\lambda_{A_T}+P\left(X_{t_y,f_y-1,y}\right)\lambda_{A_{FT}}-P\left(X_{t_y,f_y,y}\right)\left(\lambda_{A_T}+\lambda_{A_{FT}}\right)\ , \qquad (5.64)
\end{aligned}$$

$$\begin{aligned}
P\left(X_{t_y,f_y,y}|X_{t,f,x}:t_y+f_y<w_A\right) &= P\left(X_{t_y,f_y,y}\right) \\
&= e^{-\lambda_{A_T}y}\frac{\left(\lambda_{A_T}y\right)^{t_y}}{t_y!}e^{-\lambda_{A_{FT}}y}\frac{\left(\lambda_{A_{FT}}y\right)^{f_y}}{f_y!}\ . \qquad (5.65)
\end{aligned}$$

For the $t_y+f_y=w_A$ situation, we again just add back in the last term of (5.64) and solve:

$$\begin{aligned}
&P\left(X_{t_y,f_y,y}|X_{t,f,x}:t_y+f_y=w_A\right)= \\
&\quad \lambda_{A_T}^{t_y}\lambda_{A_{FT}}^{f_y}\binom{w_A}{t_y}\frac{1}{\left(\lambda_{A_T}+\lambda_{A_{FT}}\right)^{w_A}}\frac{\gamma(w_A,\left(\lambda_{A_T}+\lambda_{A_{FT}}\right)y)}{\Gamma(w_A)}\ . \qquad (5.66)
\end{aligned}$$

So then we have

$$P\left(\mathcal{T}_{t_y}|\mathcal{W}_{\mathcal{A}}=w_A\right) = \sum_{f_y=0}^{w_A-t_y-1} e^{-(\lambda_{A_T}+\lambda_{A_{FT}})y}\frac{(\lambda_{A_T}y)^{t_y}}{t_y!}\frac{(\lambda_{A_{FT}}y)^{f_y}}{f_y!} + \lambda_{A_T}^{t_y}\lambda_{A_{FT}}^{w_A-t_y}\binom{w_A}{t_y}\frac{\gamma(w_A,(\lambda_{A_T}+\lambda_{A_{FT}})\,y)}{\Gamma(w_A)\,(\lambda_{A_T}+\lambda_{A_{FT}})^{w_A}}\ , \tag{5.67}$$

which matches (4.86) when $y = 1 - x$. We see that we also have

$$P\left(\mathcal{T}_{t_y\geq\xi}|\mathcal{W}_{\mathcal{A}}=w_A\right) = \sum_{t_y=\xi}^{w_A-1}\sum_{f_y=0}^{w_A-t_y-1} e^{-(\lambda_{A_T}+\lambda_{A_{FT}})y}\frac{(\lambda_{A_T}y)^{t_y}}{t_y!}\frac{(\lambda_{A_{FT}}y)^{f_y}}{f_y!} + \sum_{t_y=\xi}^{w_A}\lambda_{A_T}^{t_y}\lambda_{A_{FT}}^{w_A-t_y}\binom{w_A}{t_y}\frac{\gamma(w_A,(\lambda_{A_T}+\lambda_{A_{FT}})\,y)}{\Gamma(w_A)\,(\lambda_{A_T}+\lambda_{A_{FT}})^{w_A}}\ , \tag{5.68}$$

and

$$E\left[t_y|\mathcal{W}_{\mathcal{A}}=w_A\right] = \sum_{t_y=0}^{w_A} t_y P\left(\mathcal{T}_{t_y}|\mathcal{W}_{\mathcal{A}}=w_A\right)\ . \tag{5.69}$$

Other probabilities are of importance as well. We'll now look at the probability of mission success and mission failure.

5.6 Probability of Mission Success

Mission success is defined as attacking at least a pre-specified number, m of targets. So we simply have a repeat of a previous probability which can be expressed in two ways:

$$\begin{aligned}
P\left(X_{\geq m, f\geq 0, x}\right) &= \sum_{t=m}^{w}\sum_{f=0}^{w-t} P\left(X_{t,f,x}\right) \ , \\
&= \sum_{t=m}^{w-1}\sum_{f=0}^{w-t-1} P\left(X_{t,f,x} : t+f < w\right) + \\
&\quad \sum_{t=m}^{w} P\left(X_{t,w-t,x} : t+f = w\right) \ , \qquad (5.70) \\
&= 1 - \sum_{t=0}^{m-1}\sum_{f=0}^{w-t} P\left(X_{t,f,x}\right) \ , \\
&= 1 - \sum_{t=0}^{m-1}\sum_{f=0}^{w-t-1} P\left(X_{t,f,x} : t+f < w\right) - \\
&\quad \sum_{t=0}^{m-1} P\left(X_{t,w-t,x} : t+f = w\right) \ . \qquad (5.71)
\end{aligned}$$

We have already computed the probability of mission success for Scenario 1 (since there is only one target). It is simply Equation (5.24). For Scenario 2, we utilize (A.5) and (A.6) to determine

$$\begin{aligned}
P\left(X_{\geq m, f\geq 0, x}\right) &= \sum_{t=m}^{w-1}\sum_{f=0}^{w-t-1} \frac{(\lambda_{A_T}x)^t}{t!}\frac{(\lambda_{A_{FT}}x)^f}{f!} e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x} + \\
&\quad \sum_{t=m}^{w} \lambda_{A_T}^t \lambda_{A_{FT}}^{w-t} \binom{w}{t} \frac{\gamma(w, (\lambda_{A_T}+\lambda_{A_{FT}})\,x)}{\Gamma(w)\,(\lambda_{A_T}+\lambda_{A_{FT}})^w} \ . \qquad (5.72)
\end{aligned}$$

or equivalently

$$\begin{aligned}
P\left(X_{\geq m, f\geq 0, x}\right) &= 1 - \sum_{t=0}^{m-1}\sum_{f=0}^{w-t-1} \frac{(\lambda_{A_T}x)^t}{t!}\frac{(\lambda_{A_{FT}}x)^f}{f!} e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x} - \\
&\quad \sum_{t=0}^{m-1} \lambda_{A_T}^t \lambda_{A_{FT}}^{w-t} \binom{w}{t} \frac{\gamma(w, (\lambda_{A_T}+\lambda_{A_{FT}})\,x)}{\Gamma(w)\,(\lambda_{A_T}+\lambda_{A_{FT}})^w} \ , \qquad (5.73)
\end{aligned}$$

5.7 Probability of Mission Failure

Mission failure is the complement of mission success;

$$
\begin{aligned}
P_{mf}(x) &= 1 - P_{ms} = 1 - \left(1 - P\left(X_{\leq m-1, f\geq 0, x}\right)\right) = P\left(X_{\leq m-1, f\geq 0, x}\right) \ , \\
&= \sum_{t=0}^{m-1} \sum_{f=0}^{w-t} P\left(X_{t,f,x}\right) \ , \\
&= \sum_{t=0}^{m-1} \sum_{f=0}^{w-t-1} P\left(X_{t,f,x} : t+f < w\right) + \sum_{t=0}^{m-1} P\left(X_{t,w-t,x} : t+f = w\right) \ . \qquad (5.74)
\end{aligned}
$$

For Scenario 1 we have

$$
P_{mf}(x) = 1 - \sum_{f=0}^{w-2} P_{TR} x e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} - \frac{w P_{TR}}{\lambda_{A_{FT}}} \frac{\gamma(w, \lambda_{A_{FT}} x)}{\Gamma(w)} \ . \qquad (5.75)
$$

For Scenario 2 we have

$$
\begin{aligned}
P_{mf}(x) &= \sum_{t=0}^{m-1} \sum_{f=0}^{w-t-1} \frac{(\lambda_{A_T} x)^t}{t!} \frac{(\lambda_{A_{FT}} x)^f}{f!} e^{-(\lambda_{A_T} + \lambda_{A_{FT}})x} + \\
&\quad \sum_{t=0}^{m-1} \lambda_{A_T}^t \lambda_{A_{FT}}^{w-t} \binom{w}{t} \frac{\gamma(w, (\lambda_{A_T} + \lambda_{A_{FT}}) x)}{\Gamma(w) (\lambda_{A_T} + \lambda_{A_{FT}})^w} \ . \qquad (5.76)
\end{aligned}
$$

Now that we have the probabilities of various combinations of target and false target attacks, we can look at probabilities which are also important, especially to commanders in the field. These probabilities involve not just target attacks but target kills.

5.8 Target Kills

Once we have the probability of target attacks, we can then incorporate the probability of killing a target. Similar to our notation for events involving TA's, we define the following events involving target kills (TK).

1. $\mathcal{K}$: Target Kill
2. $\mathcal{K}_{t_k,(\cdot)}$ t_k TK's in the space represented by $(\cdot)$ (usually the normalized time, x)

Define P_k as the probability of a target kill (TK) given a target attack and $P(\mathcal{K}_{t_k,x}|\mathcal{T}_{t,x})$ as the probability of exactly t_k target kills in x given there were exactly t target attacks in x. When $P_k < 1$ this probability takes the form of a binomial distribution:

$$P(\mathcal{K}_{t_k,x}|\mathcal{T}_{t,x}) = \binom{t}{t_k}(P_k)^{t_k}(1-P_k)^{(t-t_k)} \ . \tag{5.77}$$

However, when $P_k = 1$;

$$\begin{aligned} P(\mathcal{K}_{t_k,x}|\mathcal{T}_{t,x}) &= 1 : P_k = 1, t_k = t \ , && (5.78)\\ P(\mathcal{K}_{t_k,x}|\mathcal{T}_{t,x}) &= 0 : P_k = 1, t_k \neq t \ . && (5.79)\end{aligned}$$

We assume in this research $P_k < 1$.

We can write the equation for the probability of exactly t_k TK's:

$$\begin{aligned} P(\mathcal{K}_{t_k,x}) &= P(\mathcal{T}_{t\geq t_k,x})P(\mathcal{K}_{t_k,x}|\mathcal{T}_{t,x}) \ , \\ &= P(\mathcal{T}_{t\geq t_k,x})\binom{t}{t_k}(P_k)^{t_k}(1-P_k)^{(t-t_k)} \ . && (5.80)\end{aligned}$$

Recall that $P(\mathcal{T}_{t\geq t_k,x})$ involves a summation over t. In equation (5.80) (and any other time we see such a combination of terms), the terms after the $P(\mathcal{T}_{t\geq t_k,x})$ term are included in that summation. Now if we want to compute the probability of at least t_k TK's in x, we have

$$P(\mathcal{K}_{t_k\geq k,x}) = \sum_{t_k=k}^{w} P(\mathcal{T}_{t\geq t_k,x})\binom{t}{t_k}(P_k)^{t_k}(1-P_k)^{(t-t_k)} \ . \tag{5.81}$$

When computing these probabilities in A_{m} we have

$$P\left(\mathcal{K}_{t_{k_y}}|\mathcal{W}_{\mathcal{A}}=w_A\right)=\frac{\sum_{t_y=t_{k_y}}^{w_A}\sum_{t=0}^{w-w_A}\sum_{f_y=0}^{w_A-t_y}P\left(X_{t_y,f_y,y}|X_{t,w-w_A-t,x}\right)P\left(X_{t,w-w_A-t,x}\right)\binom{t_y}{t_{k_y}}\left(P_k\right)^{t_{k_y}}\left(1-P_k\right)^{t_y-t_{k_y}}}{\sum_{t=0}^{w-w_A}P\left(X_{t,w-w_A-t,x}\right)}, \tag{5.82}$$

$$P\left(\mathcal{K}_{t_{k_y}\geq k}|\mathcal{W}_{\mathcal{A}}=w_A\right)=\frac{\sum_{t_{k_y}=k}^{w_A}\sum_{t_y=t_{k_y}}^{w_A}\sum_{t=0}^{w-w_A}\sum_{f_y=0}^{w_A-t_y}P\left(X_{t_y,f_y,y}|X_{t,w-w_A-t,x}\right)P\left(X_{t,w-w_A-t,x}\right)\binom{t_y}{t_{k_y}}\left(P_k\right)^{t_{k_y}}\left(1-P_k\right)^{t_y-t_{k_y}}}{\sum_{t=0}^{w-w_A}P\left(X_{t,w-w_A-t,x}\right)}. \tag{5.83}$$

So, for example, in Scenario 2 we see that the probability of exactly t_k kills is

$$\begin{aligned}P(\mathcal{K}_{t_k,x}) &= \sum_{t=t_k}^{w-1}\sum_{f=0}^{w-t-1}\frac{(\lambda_{A_T}x)^t}{t!}\frac{(\lambda_{A_{FT}}x)^f}{f!}e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x}\binom{t}{t_k}(P_k)^{t_k}(1-P_k)^{(t-t_k)}+\\ &\quad \sum_{t=t_k}^{w}\lambda_{A_T}^{t}\lambda_{A_{FT}}^{(w-t)}\binom{w}{t}\frac{\gamma(w,(\lambda_{A_T}+\lambda_{A_{FT}})x)}{\Gamma(w)(\lambda_{A_T}+\lambda_{A_{FT}})^w}\binom{t}{t_k}(P_k)^{t_k}(1-P_k)^{(t-t_k)}\ .\end{aligned} \tag{5.84}$$

Recall our convention that any time the summation superscript is less than the subscript, the summation is equal to zero.

Equation (5.84) would be used in any situations requiring the expected number of TK's in x, namely

$$E[t_k]=\sum_{t_k=0}^{w}t_kP\left(\mathcal{K}_{t_k,x}\right)\ . \tag{5.85}$$

For Scenario 2, we have

$$\begin{aligned} E[t_k] &= \sum_{t_k=0}^{w-1}\sum_{t=t_k}^{w-1}\sum_{f=0}^{w-t-1}\left\{t_k\frac{(\lambda_{A_T}x)^t}{t!}\frac{(\lambda_{A_{FT}}x)^f}{f!}e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x}\binom{t}{t_k}P_k^{t_k}(1-P_k)^{t-t_k}\right\}+ \\ & \sum_{t_k=0}^{w}\sum_{t=t_k}^{w}\left\{t_k\lambda_{A_T}^t\lambda_{A_{FT}}^{w-t}\binom{w}{t}\frac{\gamma(w,(\lambda_{A_T}+\lambda_{A_{FT}})x)}{\Gamma(w)(\lambda_{A_T}+\lambda_{A_{FT}})^w}\binom{t}{t_k}P_k^{t_k}(1-P_k)^{t-t_k}\right\}. \end{aligned} \tag{5.86}$$

We also see that

$$\begin{aligned} P(\mathcal{K}_{t_k\geq k,x}) &= \sum_{t_k=k}^{w-1}\sum_{t=t_k}^{w-1}\sum_{f=0}^{w-t-1}\left\{\frac{(\lambda_{A_T}x)^t}{t!}\frac{(\lambda_{A_{FT}}x)^f}{f!}e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x}\binom{t}{t_k}P_k^{t_k}(1-P_k)^{t-t_k}\right\}+ \\ & \sum_{t_k=k}^{w}\sum_{t=t_k}^{w}\left\{\lambda_{A_T}^t\lambda_{A_{FT}}^{w-t}\binom{w}{t}\frac{\gamma(w,(\lambda_{A_T}+\lambda_{A_{FT}})x)}{\Gamma(w)(\lambda_{A_T}+\lambda_{A_{FT}})^w}\binom{t}{t_k}P_k^{t_k}(1-P_k)^{t-t_k}\right\} \end{aligned} \tag{5.87}$$

Now when looking in $\mathsf{A_m}$ we have for Scenario 2

$$\begin{aligned} & P\left(\mathcal{K}_{t_{k_y}}|\mathcal{W}_{\mathcal{A}}=w_A\right)= \\ & \sum_{t_y=t_{k_y}}^{w_A-1}\sum_{f_y=0}^{w_A-t_y-1}\frac{(\lambda_{A_T}y)^{t_y}}{t_y!}\frac{(\lambda_{A_{FT}}y)^{f_y}}{f_y!}e^{-(\lambda_{A_T}+\lambda_{A_{FT}})y}\binom{t_y}{t_{k_y}}(P_k)^{t_{k_y}}(1-P_k)^{t_y-t_{k_y}}+ \\ & \sum_{t_y=t_{k_y}}^{w_A}\lambda_{A_T}^{t_y}\lambda_{A_{FT}}^{w_A-t_y}\binom{w_A}{t_y}\frac{\gamma(w_A,(\lambda_{A_T}+\lambda_{A_{FT}})\,y)}{\Gamma(w_A)\,(\lambda_{A_T}+\lambda_{A_{FT}})^{w_A}}\binom{t_y}{t_{k_y}}(P_k)^{t_{k_y}}(1-P_k)^{t_y-t_{k_y}}\quad , \end{aligned} \tag{5.88}$$

$$\begin{aligned} & P\left(\mathcal{K}_{t_{k_y}\geq k}|\mathcal{W}_{\mathcal{A}}=w_A\right)= \\ & \sum_{t_{k_y}=k}^{w_A-1}\sum_{t_y=t_{k_y}}^{w_A-1}\sum_{f_y=0}^{w_A-t_y-1}\frac{(\lambda_{A_T}y)^{t_y}}{t_y!}\frac{(\lambda_{A_{FT}}y)^{f_y}}{f_y!}e^{-(\lambda_{A_T}+\lambda_{A_{FT}})y}\binom{t_y}{t_{k_y}}(P_k)^{t_{k_y}}(1-P_k)^{t_y-t_{k_y}}+ \\ & \sum_{t_{k_y}=k}^{w_A}\sum_{t_y=t_{k_y}}^{w_A}\lambda_{A_T}^{t_y}\lambda_{A_{FT}}^{w_A-t_y}\binom{w_A}{t_y}\frac{\gamma(w_A,(\lambda_{A_T}+\lambda_{A_{FT}})\,y)}{\Gamma(w_A)\,(\lambda_{A_T}+\lambda_{A_{FT}})^{w_A}}\binom{t_y}{t_{k_y}}(P_k)^{t_{k_y}}(1-P_k)^{t_y-t_{k_y}}\quad , \end{aligned} \tag{5.89}$$

and

$$E\left[\mathcal{K}_{t_{k_y}}|\mathcal{W}_{\mathcal{A}}=w_A\right] = \sum_{t_{k_y}=0}^{w_A}t_{k_y}P\left(\mathcal{K}_{t_{k_y}}|\mathcal{W}_{\mathcal{A}}=w_A\right)\ . \tag{5.90}$$

Which we see is the same as Equation (5.85) with some variable replacements. This is due to the independent and stationary increments associated with the Poisson processes inherent in Scenario 2.

We have used Scenarios 1 and 2 to show detailed development of the various probabilities using the Markov chain model. To be complete, we will examine the remaining scenarios; but, as we did for the sequential events method, we will focus on the development of the important probability of exactly t TA and f FTA. Once this is done, we will summarize that probability for each scenario. Then we can conduct a sensitivity analysis on various parameters. The parameters λ_T and λ_{FT} are determined by the battle space. These parameters represent the expected density of targets and false targets, respectively. However, the following parameters are determined by equipment investment and operational considerations; P_{TR}, P_{FTR}, P_k, w.

5.9 Scenario 3: Markov Chain Approach

For Scenario 3 the evaluation of the instantaneous transition probabilities is determined as follows:

$$\begin{aligned} P_{\mathcal{T}_t}(x, \Delta x) &= P\left\{\mathcal{T}_{1,\Delta x} | \mathcal{T}_{t-1,x}\right\} && (5.91) \\ &= \frac{P\left\{\mathcal{T}_{t,x} | \mathcal{T}_{1,\Delta x}\right\} P\left\{\mathcal{T}_{1,\Delta x}\right\}}{P\left\{\mathcal{T}_{t,x}\right\}} && (5.92) \end{aligned}$$

Since there are N targets uniformly distributed, then we have (assuming $N\Delta x \ll 1$)

$$\begin{aligned} P\left\{\mathcal{T}_{1,\Delta x}\right\} &= P_{TR} N \Delta x \ , && (5.93) \\ P\left\{\mathcal{T}_{t,x}\right\} &= \sum_{i=t}^{N} \binom{N}{i} x^i (1-x)^{N-i} \binom{i}{t} P_{TR}^t (1-P_{TR})^{i-t} \ . && (5.94) \end{aligned}$$

We use the truncated binomial conversion on the last equation to obtain

$$P\left\{\mathcal{T}_{t,x}\right\} = \binom{N}{t} (P_{TR} x)^t (1 - P_{TR} x)^{N-t} \ . \qquad (5.95)$$

Similarly,

$$P\left\{\mathcal{T}_{t,x}|\mathcal{T}_{1,\Delta x}\right\} = \sum_{i=t}^{N-1}\binom{N-1}{i}x^i(1-x)^{N-1-i}\binom{i}{t}P_{TR}^t(1-P_{TR})^{i-t} \ , \tag{5.96}$$

$$= \binom{N-1}{t}\left(P_{TR}x\right)^t\left(1-P_{TR}x\right)^{N-1-t} \ . \tag{5.97}$$

Therefore,

$$P_{\mathcal{T}_t}(x,\Delta x) = \frac{\binom{N-1}{t}\left(P_{TR}x\right)^t\left(1-P_{TR}x\right)^{N-1-t}NP_{TR}\Delta x}{\binom{N}{t}\left(P_{TR}x\right)^t\left(1-P_{TR}x\right)^{N-t}} \ , \tag{5.98}$$

$$= \frac{\binom{N-1}{t}NP_{TR}}{\binom{N}{t}\left(1-P_{TR}x\right)}\Delta x \ , \tag{5.99}$$

$$= \frac{(N-t)P_{TR}}{(1-P_{TR}x)}\Delta x \ . \tag{5.100}$$

Recall that $P_{\mathcal{T}_t}(x,\Delta x)$ is the probability of a TA occurring after t TA's have occurred by x.

Because of the independent and stationary increments of the Poisson process, $P_{\mathcal{F}_f}(x,\Delta x)$ is a constant;

$$P_{\mathcal{F}_f}(x,\Delta x) = \lambda_{A_{FT}}\Delta x \ . \tag{5.101}$$

Therefore

$$P_{\mathcal{T}_t}(x) = \lim_{\Delta x\to 0}\frac{P_{\mathcal{T}_t}(x,\Delta x)}{\Delta x} = \frac{(N-t)P_{TR}}{(1-P_{TR}x)} \ , \tag{5.102}$$

$$P_{\mathcal{F}_f}(x) = \lim_{\Delta x\to 0}\frac{P_{\mathcal{F}_f}(x,\Delta x)}{\Delta x} = \lambda_{A_{FT}} \ . \tag{5.103}$$

Making

$$
\begin{aligned}
\dot{P}\left(X_{t,f,x} : t+f < w\right) &= \frac{(N-(t-1))P_{TR}}{(1-P_{TR}x)} P\left(X_{t-1,f,x} : t+f < w\right) + \\
&\quad \lambda_{A_{FT}} P\left(X_{t,f-1,x} : t+f < w\right) - \\
&\quad \left(\frac{(N-t)P_{TR}}{(1-P_{TR}x)} + \lambda_{A_{FT}}\right) P\left(X_{t,f,x} : t+f < w\right) \ , \quad (5.104) \\
\dot{P}\left(X_{t,f,x} : t+f = w\right) &= \frac{(N-(t-1))P_{TR}}{(1-P_{TR}x)} P\left(X_{t-1,f,x} : t+f < w\right) + \\
&\quad \lambda_{A_{FT}} P\left(X_{t,f-1,x} : t+f < w\right) \ . \quad (5.105)
\end{aligned}
$$

Again, whenever a subscript is less than zero, that probability is zero. We solve these differential equations recursively. That is, we start with state $(i = 0, j = 0)$ incrementing the number of FTA's (j) for the given i till we get to $j = f$, then increment the i. We continue this until we get to $(i = t, j = f)$. The author used variation of parameters to solve the differential equations whenever $i + j < w$ and determined that

$$
\begin{aligned}
& P\left(X_{t,f,x} : t+f < w\right) = \\
& \quad \prod_{i=0}^{t-1}(N-i)\frac{(P_{TR}x)^t}{t!}\left(1-P_{TR}x\right)^{N-t} e^{-\lambda_{A_{FT}}x}\frac{(\lambda_{A_{FT}}x)^f}{f!} \ , \quad (5.106) \\
& = \binom{N}{t}(P_{TR}x)^t\left(1-P_{TR}x\right)^{N-t} e^{-\lambda_{A_{FT}}x}\frac{(\lambda_{A_{FT}}x)^f}{f!} \ . \quad (5.107)
\end{aligned}
$$

See Appendix E for the derivation of (5.107).

By substituting (5.107) into (5.105) we obtain the derivative for the subsequent $t + f = w$ probability,

$$
\begin{aligned}
\dot{P}\left(X_{t,f,x} : t+f = w\right) = \quad & \prod_{i=0}^{t-1}(N-i)P_{TR}^t\left(1-P_{TR}x\right)^{N-t} \\
& \times e^{-\lambda_{A_{FT}}x}\lambda_{A_{FT}}^f x^{t+f-1}\left(\frac{t+f}{t!f!}\right) \ , \quad (5.108)
\end{aligned}
$$

$$
= \binom{N}{t}(P_{TR}x)^t\left(1-P_{TR}x\right)^{N-t} e^{-\lambda_{A_{FT}}x}\frac{(\lambda_{A_{FT}}x)^f}{f!}\left(\frac{1}{x}t+\frac{1}{x}f\right) \quad (5.109)
$$

For notational convenience and to show similarities between scenarios, define

$$\mathbb{A}_u = P_{TR} x \ , \tag{5.110}$$

$$\mathbb{A}'_u = P_{TR} \ , \tag{5.111}$$

$$\mathbb{B}_p = \lambda_{A_{FT}} x \ , \tag{5.112}$$

$$\mathbb{B}'_p = \lambda_{A_{FT}} \ . \tag{5.113}$$

We then have

$$P_{\mathcal{T}_t}(x) = \frac{(N-t)\,\mathbb{A}'_u}{(1-\mathbb{A}_u)} \ , \tag{5.114}$$

$$P_{\mathcal{F}_f}(x) = \mathbb{B}'_p \ , \tag{5.115}$$

and

$$P\left(X_{t,f,x} : t+f < w\right) = \binom{N}{t} \mathbb{A}_u^t \,(1-\mathbb{A}_u)^{N-t}\, e^{-\mathbb{B}_p} \frac{\mathbb{B}_p^f}{f!} \ , \tag{5.116}$$

$$\dot{P}\left(X_{t,f,x} : t+f = w\right) = \binom{N}{t} \mathbb{A}_u^t \,(1-\mathbb{A}_u)^{N-t}\, e^{-\mathbb{B}_p} \frac{\mathbb{B}_p^f}{f!} \left(\frac{\mathbb{A}'_u}{\mathbb{A}_u} t + \frac{\mathbb{B}'_p}{\mathbb{B}_p} f\right) \ . \tag{5.117}$$

5.10 *Scenario 4: Markov Chain Approach*

For Scenario 4 the evaluation of the instantaneous transition probabilities is as follows:

$$P_{\mathcal{T}_t}(x, \Delta x) = P\left\{\mathcal{T}_{1,\Delta x} | \mathcal{T}_{t,x}\right\} \tag{5.118}$$

$$= \frac{P\left\{\mathcal{T}_{t,x} | \mathcal{T}_{1,\Delta x}\right\} P\left\{\mathcal{T}_{1,\Delta x}\right\}}{P\left\{\mathcal{T}_{t,x}\right\}} \tag{5.119}$$

Since there are N targets uniformly distributed, then we have the same results as in Scenario 3

$$P_{\mathcal{T}_t}(x, \Delta x) = \frac{(N-t)P_{TR}}{(1-P_{TR}x)}\Delta x \ , \tag{5.120}$$

$$\lim_{\Delta x \to 0} \frac{P_{\mathcal{T}_t}(x, \Delta x)}{\Delta x} = \frac{(N-t)P_{TR}}{(1-P_{TR}x)} \ . \tag{5.121}$$

Using the similarity between a TA and FTA where

$$N \sim M, \tag{5.122}$$

$$t \sim f \ , \tag{5.123}$$

$$P_{TR} \sim (1 - P_{FTR}) \ , \tag{5.124}$$

makes

$$P_{\mathcal{F}_f}(x, \Delta x) = \frac{(M-f)(1-P_{FTR})}{(1-(1-P_{FTR})x)}\Delta x \ , \tag{5.125}$$

$$P_{\mathcal{F}_f}(x) = \lim_{\Delta x \to 0} \frac{P_{\mathcal{F}_f}(x, \Delta x)}{\Delta x} = \frac{(M-f)(1-P_{FTR})}{(1-(1-P_{FTR})x)} \ . \tag{5.126}$$

Making

$$\begin{aligned}
&\dot{P}\left(X_{t,f,x} : t+f < w\right) = \\
&\quad \frac{(N-(t-1))P_{TR}}{(1-P_{TR}x)} P\left(X_{t-1,f,x} : t+f < w\right) + \\
&\quad \frac{(M-(f-1))(1-P_{FTR})}{(1-(1-P_{FTR})x)} P\left(X_{t,f-1,x} : t+f < w\right) - \\
&\quad \left(\frac{(N-t)P_{TR}}{(1-P_{TR}x)} + \frac{(M-f)(1-P_{FTR})}{(1-(1-P_{FTR})x)}\right) P\left(X_{t,f,x} : t+f < w\right) \ ,
\end{aligned} \tag{5.127}$$

$$\begin{aligned}
&\dot{P}\left(X_{t,f,x} : t+f = w\right) = \\
&\quad \frac{(N-(t-1))P_{TR}}{(1-P_{TR}x)} P\left(X_{t-1,f,x} : t+f < w\right) + \\
&\quad \frac{(M-(f-1))(1-P_{FTR})}{(1-(1-P_{FTR})x)} P\left(X_{t,f-1,x} : t+f < w\right) \ .
\end{aligned} \tag{5.128}$$

Again, whenever a subscript is less than zero, that probability is zero. We solve these differential equations recursively. That is, we start with state $(i = 0, j = 0)$ incrementing the number of FTA's (j) for the given i until we get to $j = f$, then increment the i. We continue this until we get to $(i = t, j = f)$. The author used variation of parameters to solve the differential equations whenever $i + j < w$ and determined that

$$P\left(X_{t,f,x} : t + f < w\right) = \prod_{i=0}^{t-1}(N-i)\frac{(P_{TR}x)^t}{t!}\left(1 - P_{TR}x\right)^{N-t} \times$$
$$\prod_{j=0}^{f-1}(M-j)\frac{((1-P_{FTR})x)^f}{f!}\left(1-(1-P_{FTR})x\right)^{M-f} \quad , \tag{5.129}$$

$$P\left(X_{t,f,x} : t + f < w\right) = \binom{N}{t}(P_{TR}x)^t(1 - P_{TR}x)^{N-t}\times$$
$$\binom{M}{f}((1-P_{FTR})x)^f(1-(1-P_{FTR})x)^{M-f} \quad . \tag{5.130}$$

So then for $t + f = w$

$$\dot{P}\left(X_{t,f,x} : t + f = w\right) = \prod_{i=0}^{t-1}(N-i)P_{TR}^t\left(1 - P_{TR}x\right)^{N-t} \times$$
$$\prod_{j=0}^{f-1}(M-j)\left((1-P_{FTR})\right)^f\left(1-(1-P_{FTR})x\right)^{M-f} x^{t+f-1}\left(\frac{t+f}{t!f!}\right) \tag{5.131}$$

$$\dot{P}\left(X_{t,f,x} : t + f = w\right) = \binom{N}{t}(P_{TR}x)^t(1 - P_{TR}x)^{N-t}\times$$
$$\binom{M}{f}((1-P_{FTR})x)^f(1-(1-P_{FTR})x)^{M-f}\left(\frac{1}{x}t + \frac{1}{x}f\right) \quad . \tag{5.132}$$

Similar to Scenario 3, we utilize

$$\mathbb{A}_u = P_{TR}x \ , \tag{5.133}$$
$$\mathbb{A}'_u = P_{TR} \ , \tag{5.134}$$

and define

$$\mathbb{B}_u = (1 - P_{FTR})x \ , \tag{5.135}$$

$$\mathbb{B}'_u = (1 - P_{FTR}) \ . \tag{5.136}$$

We then have the same $P_{\mathcal{T}_t}(x)$ as in Scenario 3 and in addition we have

$$P_{\mathcal{F}_f}(x) = \frac{(M-f)\mathbb{B}'(x)}{1-\mathbb{B}(x)} \ . \tag{5.137}$$

Therefore,

$$P\left(X_{t,f,x} : t+f < w\right) = \binom{N}{t} \mathbb{A}_u^t \left(1-\mathbb{A}_u\right)^{N-t} \binom{M}{f} \mathbb{B}_u^f \left(1-\mathbb{B}_u\right)^{M-f} \ , \tag{5.138}$$

$$\begin{aligned} \dot{P}\left(X_{t,f,x} : t+f = w\right) &= \binom{N}{t} \mathbb{A}_u^t \left(1-\mathbb{A}_u\right)^{N-t} \binom{M}{f} \mathbb{B}_u^f \left(1-\mathbb{B}_u\right)^{M-f} \\ &\quad \times \left(\frac{\mathbb{A}'_u}{\mathbb{A}_u} t + \frac{\mathbb{B}_u}{\mathbb{B}_u} f\right) \ . \end{aligned} \tag{5.139}$$

5.11 Scenario 5: Markov Chain Approach

For Scenario 5 the evaluation of the instantaneous transition probabilities is as follows:

$$P_{\mathcal{T}_t}(\rho, \Delta\rho) = P\left\{\mathcal{T}_{1,\Delta\rho} | \mathcal{T}_{t,\rho}\right\} \tag{5.140}$$

$$= \frac{P\left\{\mathcal{T}_{t,\rho} | \mathcal{T}_{1,\Delta\rho}\right\} P\left\{\mathcal{T}_{1,\Delta\rho}\right\}}{P\left\{\mathcal{T}_{t,\rho}\right\}} \tag{5.141}$$

Since there are N targets with a circular normal distribution we have (assuming $N\rho\Delta\rho \ll 1$)

$$
\begin{aligned}
P\left\{\mathcal{T}_{1,\Delta\rho}\right\} &= N\frac{\rho}{\sigma_T^2}e^{-\frac{\rho^2}{2\sigma_T^2}}P_{TR}\Delta\rho \;, && (5.142)\\
P\left\{\mathcal{T}_{t,\rho}|\mathcal{T}_{1,\Delta\rho}\right\} &= \sum_{i=t}^{N-1}\binom{N-1}{i}\left(1-e^{-\frac{\rho^2}{2\sigma_T^2}}\right)^i\left(e^{-\frac{\rho^2}{2\sigma_T^2}}\right)^{(N-1-i)} \times \\
&\quad \binom{i}{t}P_{TR}^t(1-P_{TR})^{i-t} \;, && (5.143)\\
P\left\{\mathcal{T}_{t,\rho}\right\} &= \sum_{i=t}^{N}\binom{N}{i}\left(1-e^{-\frac{\rho^2}{2\sigma_T^2}}\right)^i\left(e^{-\frac{\rho^2}{2\sigma_T^2}}\right)^{(N-i)} \times \\
&\quad \binom{i}{t}P_{TR}^t(1-P_{TR})^{i-t} \;. && (5.144)
\end{aligned}
$$

Using truncated binomial conversion, we obtain

$$
\begin{aligned}
P\left\{\mathcal{T}_{t,\rho}|\mathcal{T}_{1,\Delta\rho}\right\} &= \binom{N-1}{t}\left((1-e^{-\frac{\rho^2}{2\sigma_T^2}})P_{TR}\right)^t \times \\
&\quad \left[\left(1-e^{-\frac{\rho^2}{2\sigma_T^2}}\right)(1-P_{TR})+\left(e^{-\frac{\rho^2}{2\sigma_T^2}}\right)\right]^{(N-t-1)} \;, && (5.145)\\
P\left\{\mathcal{T}_{t,\rho}\right\} &= \binom{N}{t}\left(\left(1-e^{-\frac{\rho^2}{2\sigma_T^2}}\right)P_{TR}\right)^t \times \\
&\quad \left[\left(1-e^{-\frac{\rho^2}{2\sigma_T^2}}\right)(1-P_{TR})+\left(e^{-\frac{\rho^2}{2\sigma_T^2}}\right)\right]^{N-t} \;. && (5.146)
\end{aligned}
$$

Therefore,

$$
P_{\mathcal{T}_t}(\rho,\Delta\rho) = \frac{(N-t)\frac{\rho}{\sigma_T^2}e^{-\frac{\rho^2}{2\sigma_T^2}}P_{TR}\Delta\rho}{\left[1-P_{TR}\left(1-e^{-\frac{\rho^2}{2\sigma_T^2}}\right)\right]} \qquad (5.147)
$$

and for the Poisson distribution of false targets we have

$$
P_{\mathcal{F}_f}(\rho,\Delta\rho) = 2\alpha_c\pi\rho(1-P_{FTR})\rho\Delta\rho \;. \qquad (5.148)
$$

By defining

$$\mathbb{A}_c = \left(1 - e^{-\frac{\rho^2}{2\sigma_T^2}}\right) P_{TR} \ , \tag{5.149}$$

$$\mathbb{A}'_c = \frac{\rho}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} P_{TR} \ , \tag{5.150}$$

$$\mathbb{B}_{cp} = \alpha_c \pi \rho^2 (1 - P_{FTR}) \ , \tag{5.151}$$

$$\mathbb{B}'_{cp} = 2\alpha_c \pi \rho (1 - P_{FTR}) \ , \tag{5.152}$$

we can write our instantaneous transition probabilities as

$$P_{\mathcal{T}_t}(\rho) = \frac{(N-t)\mathbb{A}'_c}{[1-\mathbb{A}_c]} \ , \tag{5.153}$$

and

$$P_{\mathcal{F}_f}(\rho) = \mathbb{B}'_{cp} \ . \tag{5.154}$$

So then the Chapman-Kolmogorov equations are

$$\begin{aligned} \dot{P}\left(X_{t,f,\rho} : t+f < w\right) &= P_{\mathcal{T}_{t-1}}(\rho) P\left(X_{t-1,f,\rho} : t+f < w\right) + \\ &\quad P_{\mathcal{F}_{f-1}}(\rho) P\left(X_{t,f-1,\rho} : t+f < w\right) - \\ &\quad \left(P_{\mathcal{T}_t}(\rho) + P_{\mathcal{F}_f}(\rho)\right) P\left(X_{t,f,\rho} : t+f < w\right) \ , \end{aligned} \tag{5.155}$$

$$\begin{aligned} \dot{P}\left(X_{t,f,\rho} : t+f = w\right) &= P_{\mathcal{T}_{t-1}}(\rho) P\left(X_{t-1,f,\rho} : t+f < w\right) + \\ &\quad P_{\mathcal{F}_{f-1}}(\rho) P\left(X_{t,f-1,\rho} : t+f < w\right) \ . \end{aligned} \tag{5.156}$$

Using same method for solving the differential equation as in previous scenarios, we obtain

$$P\left(X_{t,f,\rho} : t+f < w\right) = \binom{N}{t} \mathbb{A}_c^t \left(1-\mathbb{A}_c\right)^{N-t} e^{-\mathbb{B}_{cp}} \frac{\mathbb{B}_{cp}^f}{f!} \ , \tag{5.157}$$

$$\dot{P}\left(X_{t,f,\rho} : t+f = w\right) = \binom{N}{t} \mathbb{A}_c^t \left(1-\mathbb{A}_c\right)^{N-t} e^{-\mathbb{B}_{cp}} \frac{\mathbb{B}_{cp}^f}{f!} \left(\frac{\mathbb{A}'_c}{\mathbb{A}_c} t + \frac{\mathbb{B}'_{cp}}{\mathbb{B}_{cp}} f\right) \ . \tag{5.158}$$

At this point we note the similarity in form between Scenarios 3 and 5. This similarity can be seen by comparing (5.116) and (5.117) with (5.157) and (5.158). If we could solve (5.158) in that form, we would solve the Markov formulation for Scenario 3 at the same time. All that would be required would be to make the appropriate substitutions for $\mathbb{A}$ and $\mathbb{B}$. However, this equation proved intractable in that form.

We have noted in Chapter IV that $\dot{P}\left(X_{t,f,\rho} : t+f=w\right)$ can be integrated by using binomial conversions to convert polynomials to series; however, the solution is very computationally intensive. It is also time intensive to solve (5.155) recursively (such as shown in Appendix E). Therefore, an attempt was made to solve the Chapman-Kolmogorov equation without having to use the recursive solution to the differential equations approach. The unsuccessful attempt to solve it can be found in Appendix F.

5.12 Scenario 6: Markov Chain Approach

For Scenario 6 the evaluation of the instantaneous transition probabilities is as follows:

$$P_{\mathcal{T}_t}(\rho,\Delta\rho) = P\left\{\mathcal{T}_{1,\Delta\rho}|\mathcal{T}_{t,\rho}\right\} \tag{5.159}$$

$$= \frac{P\left\{\mathcal{T}_{t,\rho}|\mathcal{T}_{1,\Delta\rho}\right\}P\left\{\mathcal{T}_{1,\Delta\rho}\right\}}{P\left\{\mathcal{T}_{t,\rho}\right\}} \tag{5.160}$$

We use the same equations for the targets as we did in Scenario 5.

$$P_{\mathcal{T}_t}(\rho,\Delta\rho) = \frac{(N-t)\frac{\rho}{\sigma_T^2}e^{-\frac{\rho^2}{2\sigma_T^2}}P_{TR}\Delta\rho}{\left[1-P_{TR}\left(1-e^{-\frac{\rho^2}{2\sigma_T^2}}\right)\right]} \tag{5.161}$$

By similarity, we have for the M false targets,

$$P_{\mathcal{F}_f}(\rho, \Delta\rho) = \frac{(M-f)\frac{\rho}{\sigma_{FT}^2} e^{-\frac{\rho^2}{2\sigma_{FT}^2}} (1 - P_{FTR})\Delta\rho}{\left[1 - (1 - P_{FTR})\left(1 - e^{-\frac{\rho^2}{2\sigma_{FT}^2}}\right)\right]} \tag{5.162}$$

By utilizing previous definitions

$$\mathbb{A}_c = \left(1 - e^{-\frac{\rho^2}{2\sigma_T^2}}\right) P_{TR} \ , \tag{5.163}$$

$$\mathbb{A}'_c = \frac{\rho}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} P_{TR} \ , \tag{5.164}$$

and defining new ones

$$\mathbb{B}_c = \left(1 - e^{-\frac{\rho^2}{2\sigma_{FT}^2}}\right) (1 - P_{FTR}) \ , \tag{5.165}$$

$$\mathbb{B}'_c = \frac{\rho}{\sigma_{FT}^2} e^{-\frac{\rho^2}{2\sigma_{FT}^2}} (1 - P_{FTR}) \ , \tag{5.166}$$

we can write our instantaneous transition probabilities as

$$P_{\mathcal{T}_t}(\rho) = \frac{(N-t)\mathbb{A}'_c}{[1 - \mathbb{A}_c]} \ , \tag{5.167}$$

and

$$P_{\mathcal{F}_f}(\rho) = \frac{(M-f)\mathbb{B}'_c}{[1 - \mathbb{B}_c]} \ , \ . \tag{5.168}$$

So then the Chapman-Kolmogorov equations are

$$\begin{aligned}
\dot{P}\left(X_{t,f,\rho} : t+f < w\right) &= P_{\mathcal{T}_{t-1}}(\rho) P\left(X_{t-1,f,\rho} : t+f < w\right) + \\
&\quad P_{\mathcal{F}_{f-1}}(\rho) P\left(X_{t,f-1,\rho} : t+f < w\right) - \\
&\quad \left(P_{\mathcal{T}_t}(\rho) + P_{\mathcal{F}_f}(\rho)\right) P\left(X_{t,f,\rho} : t+f < w\right) \ , \qquad (5.169) \\
\dot{P}\left(X_{t,f,\rho} : t+f = w\right) &= P_{\mathcal{T}_{t-1}}(\rho) P\left(X_{t-1,f,\rho} : t+f < w\right) + \\
&\quad P_{\mathcal{F}_{f-1}}(\rho) P\left(X_{t,f-1,\rho} : t+f < w\right) \ . \qquad (5.170)
\end{aligned}$$

Using same method for solving the differential equation as in previous scenarios, we obtain

$$\begin{aligned}
P\left(X_{t,f,\rho} : t+f < w\right) &= \binom{N}{t} \mathbb{A}_c^t \left(1 - \mathbb{A}_c\right)^{N-t} \binom{M}{f} \mathbb{B}_c^f \left(1 - \mathbb{B}_c\right)^{M-f} \ , \qquad (5.171) \\
\dot{P}\left(X_{t,f,\rho} : t+f = w\right) &= \binom{N}{t} \mathbb{A}_c^t \left(1 - \mathbb{A}_c\right)^{N-t} \binom{M}{f} \mathbb{B}_c^f \left(1 - \mathbb{B}_c\right)^{M-f} \\
&\quad \times \left(\frac{\mathbb{A}_c'}{\mathbb{A}_c} t + \frac{\mathbb{B}_c'}{\mathbb{B}_c} f\right) \ . \qquad (5.172)
\end{aligned}$$

We can see the similarities in form between Scenarios 4 and 6 by comparing (5.138) and (5.139) with (5.171) and (5.172). As with Scenarios 3 and 5, (5.172) proved intractable in this form. And solutions derived using the sequential events method were computationally intensive. It was much faster to solve (5.117), (5.139), (5.158), and (5.172) using numerical integration techniques. Appendix A summarizes the probability of an exact number of TA and FTA for each scenario.

VI. *Application of Main Results*

We now discuss the application of the probabilities computed in this research. We will follow a notional UCAV from design to use in the battlefield and examine the use of these probabilities at several critical stages: weapon system design, operational employment, and tactical decision making. In this discussion we confine our attention to Scenario 2, although it is possible to perform a similar analysis for the other scenarios.

6.1 System Design

When designing the system, we must balance the desire to attack real targets with the desire to minimize false targets (collateral damage). In essence, this is a tradeoff between the P_{TR} and P_{FTR} parameters in the confusion matrix. Recall that

$$\lambda_{A_T} = P_{TR}\lambda_T \ , \tag{6.1}$$

$$\lambda_{A_{FT}} = (1 - P_{FTR})\lambda_{FT} \ . \tag{6.2}$$

In addition to the parameters involving the sensor, P_{TR} and P_{FTR}, we also have the parameter which defines the effectiveness of the warhead, P_k. Each sensor has its own possible values of P_{TR} and P_{FTR} and each warhead design has a particular value of P_k. We must choose appropriate values (the appropriate sensor settings and warhead) for these parameters. We can use the probabilities developed in this research to make that determination. First, we must have some understanding of, or make assumptions about, the range of environments in which this design will be utilized. Based on intelligence or enemy doctrine, a distribution of targets and false targets is assumed. For purposes of illustration, we will assume Poisson distributions. For now we will assume $\lambda_{FT} = 20, \lambda_T = 10$, but the ratio of these parameters will be varied in subsequent sections. To start this analysis, we calculate the expected number of target kills and assume for the time being that we have ten warheads, each with $P_k = 0.7$. We have a choice of two sensors. For each sensor, we assume that the probability of target attack P_{TR} and probability of false target attack $1 - P_{FTR}$ are related to a threshold parameter, h. When the target correlation is above the threshold parameter, the object being examined is declared a target. When it is below

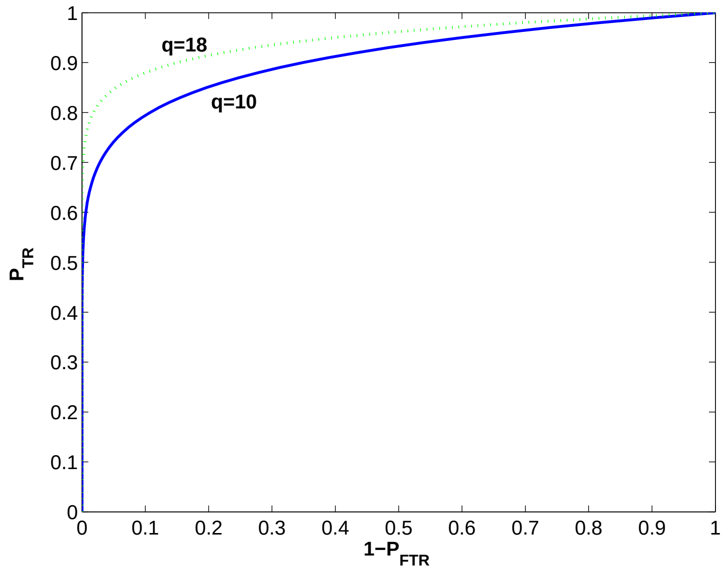

Figure 6.1 Scenario 2: ROC curves for two possible sensors

the threshold, it is declared a false target. For simplicity sake, we define P_{TR} and P_{FTR} to be related to h in the following way.

Table 6.1 Sensors

Sensor	$P(\mathcal{T})$ Eqn	$P(\mathcal{F})$ Eqn	ROC Eqn	$P_{FTR}(P_{TR})$ Eqn
Sensor 1	$P_{TR} = 1 - h$	$1 - P_{FTR} = (1-h)^{10}$	$1 - P_{FTR} = P_{TR}^{10}$	$P_{FTR} = 1 - P_{TR}^{10}$
Sensor 2	$P_{TR} = 1 - h$	$1 - P_{FTR} = (1-h)^{18}$	$1 - P_{FTR} = P_{TR}^{18}$	$P_{FTR} = 1 - P_{TR}^{18}$

So, in general, our sensors are characterized by the governing equation $P_{FTR} = 1 - P_{TR}^q$, where $q = 10, 18$. The Receiver Operating Characteristic (ROC) curve shows the relation between P_{TR} and $1 - P_{FTR}$ by plotting one as a function of the other. The ROC curves for these sensors are found in Figure 6.1. We can examine the various sensors in terms of the expected number of target kills. This produces the plots in Figure 6.2.

The designer sees from this Figure that for this scenario, and for $P_k = 0.7$ and $w = 10$, the $q = 18$ sensor does the better job. In addition, we see that at this P_k, there is no point in raising the threshold so that $P_{TR} > 0.84$. The reason is found in this sensors' ROC curve in Figure 6.1, where it can be seen that above $P_{TR} = 0.84$, the probability of false target attack increases faster than the probability of target attack. We also see from

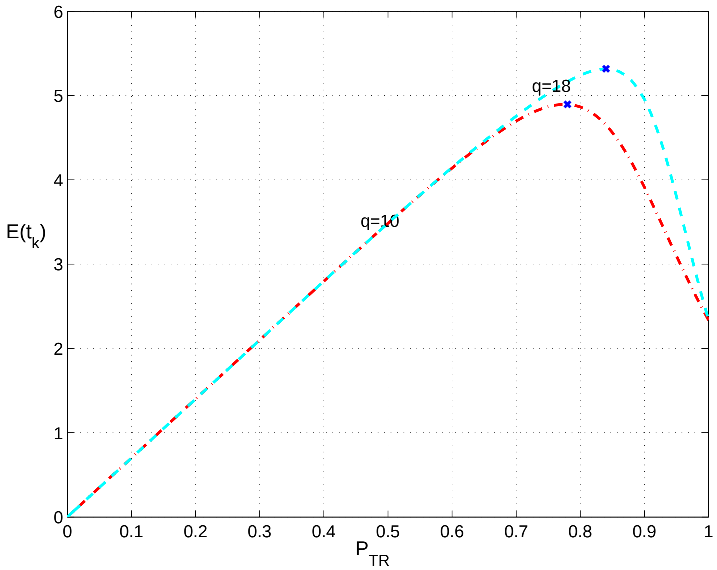

Figure 6.2 Scenario 2: Expected number of TK for two sensors, $w = 10$, $\lambda_{FT} = 20, \lambda_T = 10, P_k = 0.7, x = 1$

Figure 6.2 that if for some reason we were to limit P_{TR} to less than 0.7, then we might as well go with the $q = 10$ sensor - presumably a cheaper sensor.

We now examine the relationship between P_{TR} and P_k for the $q = 18$ sensor as shown in Figure 6.3. Figure 6.3 not only indicates that $P_{TR} = 0.84$ gives the maximum expected number of target kills, but it also shows that this value is linear (or at least nearly so) with respect to P_k. This seems to verify an expectation that $E[t_k] = P_k E[t]$.

Now we analyze the impact of the number of warheads on the expected number of TK's. Note that our previous determination of the P_{TR} which gave the maximum $E[t_k]$ was only applicable for $w = 10$. For example, with everything else the same, the P_{TR} giving the best $E[t_k]$ went up as high as $P_{TR} = 0.92$ for $w = 18$. For the rest of the analysis in this section we will set $P_{TR} = 0.9$.

The expected number of TK's for various warhead capacities is found in Figure 6.4. From this Figure we note that with more than fourteen warheads, we start to get diminishing returns. Obviously, the point at which we get diminishing returns is strongly related to λ_T and λ_{FT}. As a result of Figure 6.4, we decide to incorporate into our design a maximum of fourteen warheads.

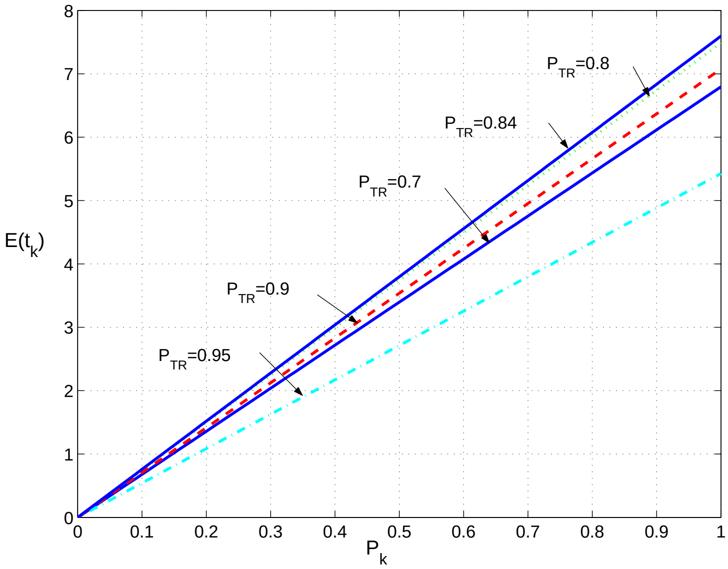

Figure 6.3 Scenario 2: Expected number of TK as a function of warhead lethality, $w = 10$, $q = 18$, $\lambda_{FT} = 20, \lambda_T = 10, x = 1$

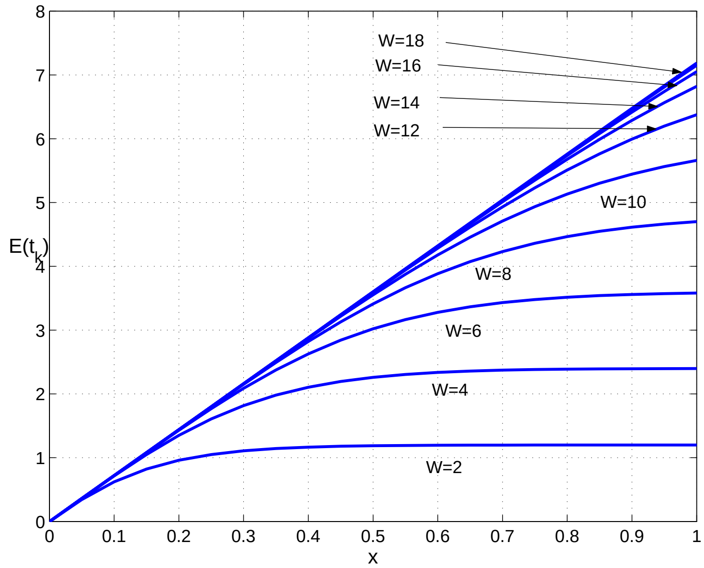

Figure 6.4 Scenario 2: Expected number of Target Kills for various warhead capacities, $\lambda_{FT} = 20, \lambda_T = 10, P_k = 0.8, P_{TR} = 0.9, q = 18$

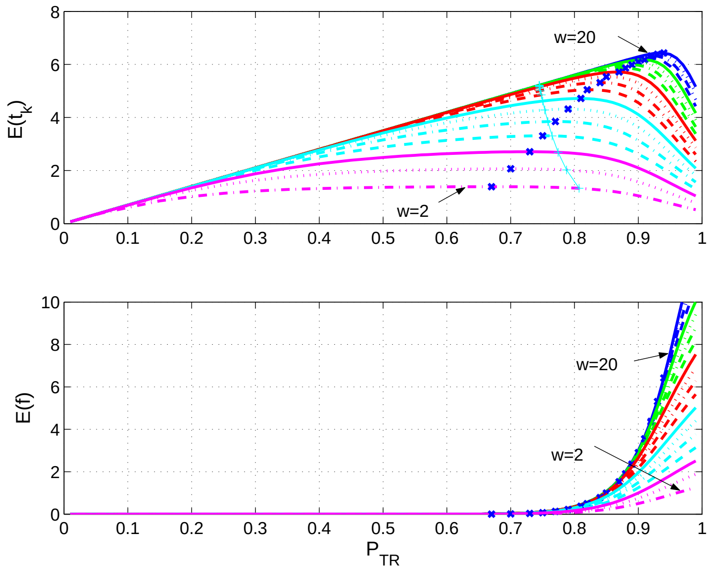

Figure 6.5 Scenario 2: Expected number of Target Kills and False Target Attacks for various warhead capacities, $\lambda_{FT} = 20,\ \lambda_T = 10,\ P_k = 0.7,\ q = 18,\ E[f] \leq 0.1$

The preceding discussion examined the number of expected target kills regardless of the number of FTA's. However, FTA's can be considered to be equivalent to collateral damage. Therefore, we want to minimize FTA's or constrain the number of FTA's to some number. We will examine the expected number of target kills for various w's ($w = 2 : 20$) and various P_{TR}'s (for a particular λ_{FT} and λ_T) with a maximum expected number of FTA's ($E[f] \leq 0.1$). This is found in Figure 6.5.

The unconstrained maximum $E[t_k]$'s are marked in Figure 6.5 by $\mathbf{x}$'s in the top plot. Not surprisingly, the maximum increases as w increases. However, there is a plateau for each curve due to the fact that we have more false targets than targets. As P_{TR} increases, the correlation threshold decreases meaning we are less discriminating when declaring a target. Therefore, we will have more FTA's. At some point, the FTA's use up too many warheads reducing the number of possible TA's.

We see that as we increase w, along with more TK's, we have more FTA's. We also see that we have more FTA's as we increase P_{TR}. We can therefore limit the number of FTA's by setting a minimum threshold level which gives a maximum P_{TR} and therefore a maximum number of FTA's for that w. On the upper plot we can then plot the $E[t_k]$ for that constrained P_{TR} and w. The result is the vertically diagonal line on the upper plot of Figure 6.5. This line represents the constraint of keeping $E[f]$ below 0.1. Anywhere to the left of this line violates the constraint. If we stick to the constraint, we notice that as we increase w we get to a point of diminishing returns. In this case, it appears there is no advantage in increasing w above $w = 12$. Therefore, a designer who is designing for this situation might decide to design for a maximum of twelve warheads. In addition, we can examine the variance for target kills.

Figure 6.6 shows the expected number of target kills, its standard deviation of the mean, and two constraint lines for the same scenario as used in Figure 6.5 except that P_k is now 0.8. One constraint line is associated with the constraint $E[f] \leq 0.1$. The other constraint displayed is $E[f] + \sigma \leq 0.1$. The second constraint will obviously be more stringent since a greater percentage of the possible number of FTA's are below the constraint line. Note that in Figure 6.6 we do not show the $E[f]$ plots. To meet the more stringent constraint on FTA's we must raise the ROC threshold thereby lowering P_{TR} which of course lowers $E[t_k]$. To take the variance into account, we could then compute $E[t_k] - \sigma$ for any given P_{TR} and w. This will not necessarily be an integer, but we can round down to the lower integer and round up to the higher integer to determine upper and lower approximations to the $E[t_k] - \sigma$ value. We can then determine the probability of obtaining at least those approximate number of target kills. This was done for the values along the two constraint lines. The result is found in Figure 6.7. This figure could be used whenever the values of $E[t_k] - \sigma$ are approximately the same for the two constraints. In that case, we would use the values from the constraint where the probability is greatest according to plots such as that found in Figure 6.7.

The design factor of $w = 12$ is valid for this particular set of λ_{FT} and λ_T. We can do similar analysis for other λ_T's and λ_{FT}'s and create a plot which, in effect, gives a design space. To make this plot we must first set a desired increment for $E[t_k]$. The first time an increase in w produces an increment less than the desired increment will be considered the point of diminishing returns. That w is then declared to be the maximum constrained w. In addition, we decided to use a ratio $r = \lambda_{FT}/\lambda_T$ which in combination with a λ_T

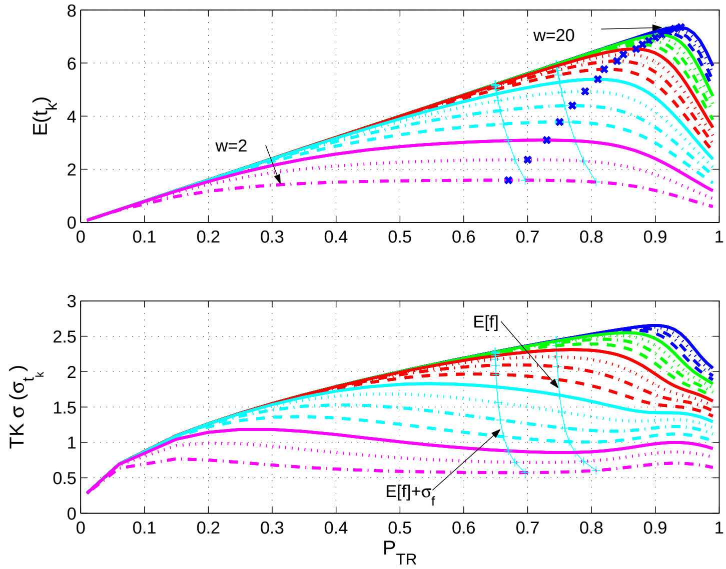

Figure 6.6 Scenario 2: Expected number of Target Kills and Standard Deviation for various warhead capacities ($w = 2 : 20$), $\lambda_{FT} = 20$, $\lambda_T = 10$, $P_k = 0.8$, $q = 18$, $E[f] \leq 0.1$

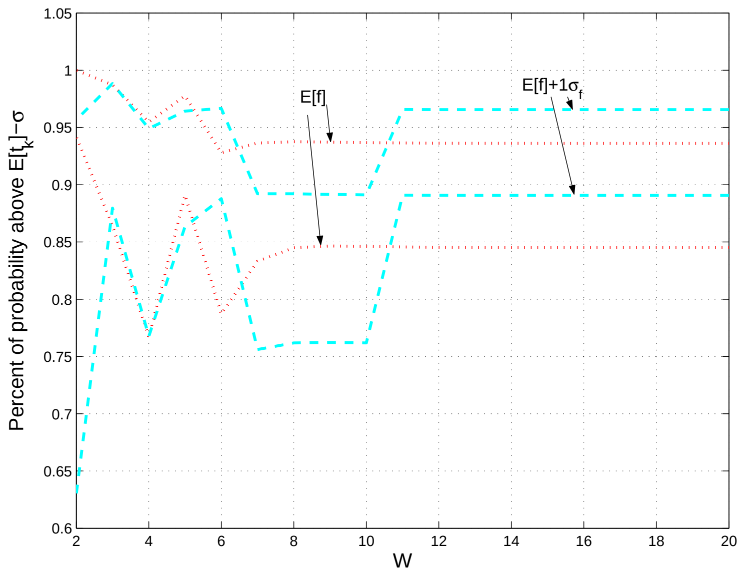

Figure 6.7 Scenario 2: Probability of at least $E[t_k] - \sigma$ for various warhead capacities ($w = 2 : 20$) and constraints, $\lambda_{FT} = 20$, $\lambda_T = 10$, $P_k = 0.8$, $q = 18$

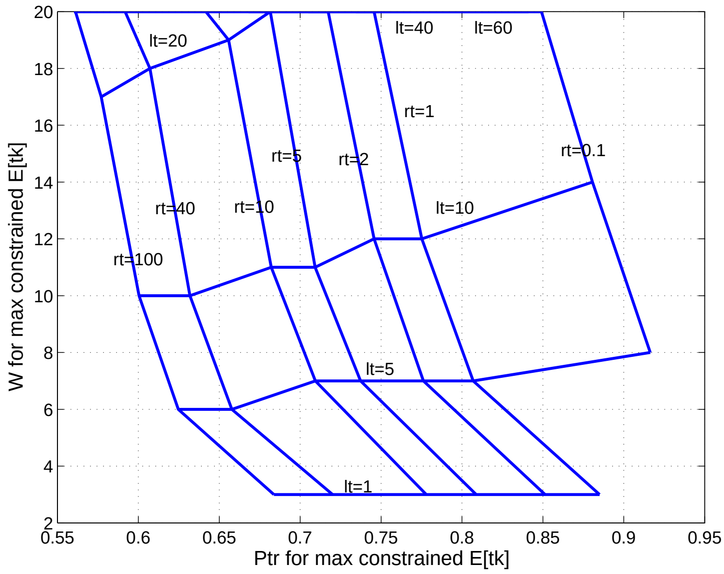

Figure 6.8 Scenario 2: Design space for maximum constrained $E[t_k]$: $P_k = 0.8$, $q = 18$, $E[f] \leq 0.1$

defines a λ_{FT}. The result is found in Figure 6.8. This figure gives the best P_{TR} and w which maximizes $E[t_k]$ while constraining $E[f] \leq 0.1$.

Once a design is chosen, we can then see how robust it is to the possible ranges of λ_T's and r's we could expect to come across. Figure 6.9 shows the expected number of TK's for various λ_T's and r's. The $\mathbf{x}$'s correspond to the constraint $E[f] \leq 0.1$. This figure shows some general rules of thumb a designer should keep in mind. We shall first conduct our analysis assuming no constraint on FTA, then examine those results when enforcing a constraint on FTA. To see the first rule of thumb, let us assume that we have correctly guessed/estimated/determined the ratio but are incorrect about λ_T. Let us say that we designed for $\lambda_T = 5$ and $r = 10$. That being the case, we would have chosen the P_{TR} which gives us point A1 in the figure. Let us then assume that in reality, unbeknownst to us, $r = 10$ but λ_T is less than 5, say $\lambda_T = 2$. So then, while we are thinking we are operating at point A1, we are actually operating at point A2. We see that the resulting $E[t_k]$ is not very different from what we would have obtained had we operated at the maximum for $\lambda_T = 2$, $r = 10$. Now let us examine the opposite situation, where we underestimate λ_T instead of overestimate it. Say that once again we think we are operating at A1 but in reality $\lambda_T = 10$ which means we are really operating at A3. We see that in this case the result is significantly different than if we would have know we were at $\lambda_T = 10,\ r = 10$ and had set P_{TR} to be on the curve's maximum. We can see from the figure, that this is true in general, it is better to overestimate λ_T than to underestimate it (given the estimate of r is correct). That is the first rule of thumb.

The second rule of thumb is determined by assuming we have correctly determined λ_T but incorrectly determined r. Using the same type of analysis, let us say that we designed for $\lambda_T = 10,\ r = 2$ (point B1) but in reality had either underestimated (in reality $r = 10$, point B2) or had overestimated (r really 0.1, point B3). We see that while overestimating r causes some loss in $E[t_k]$ compared to the optimum for that curve, underestimating creates an even bigger loss. Again, the figure indicates this is true in general. Therefore, the second rule of thumb is that it is better to overestimate the ratio (given λ_T is correct). Therefore, the more robust design would be one that is designed for the maximum λ_T and r (when designing for a given range of λ_T and r).

We now examine the rules of thumb when we include the FTA constraint. Recall the $\mathbf{x}$'s in Figure 6.9 represent the FTA constraint. For each λ_T, the $\mathbf{x}$ on the furthest right is for the minimum ratio and the $\mathbf{x}$ on the furthest left is for the maximum ratio. We see that if we underestimate either λ_T or r we would violate the constraint. Therefore, the rules of thumb hold for the constrained design as well. It should be pointed out that if the threshold level of the ATR can be changed in the field, then Figure 6.9 becomes important at the tactical level as well. Without good estimates of the distribution parameters, the operator would want to set the ATR ROC threshold to coincide with the high λ_T and high r curve (for either the unconstrained or constrained case, as desired).

For the rest of this chapter, we will assume that we are designing for $1 \le \lambda_T \le 10$ and $0.1 \le r \le 2$. Therefore, according to the rules of thumb we design for $\lambda_T = 10$ and $r = 2$.

Once we have the values we will use for P_{TR}, P_{FTR}, P_k, w, we may then want to evaluate this design further by doing some simulations. Whenever simulations are being used, the question usually arises as to the validity of the simulation. A partial validation can be accomplished using the probabilities described in this research. The designer sets

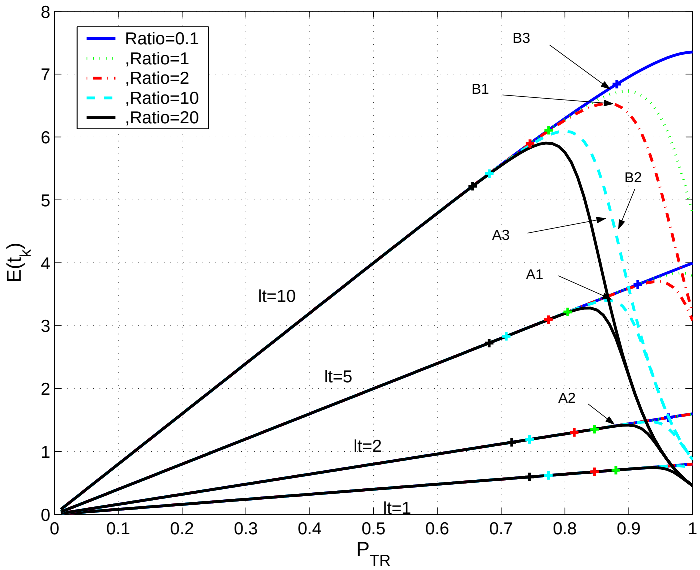

Figure 6.9 Scenario 2: Environment Robustness $E[t_k]$: $w = 12$, $P_k = 0.8$, $q = 18$, $E[f] \leq 0.1$

up the simulation to correspond with Scenario 2 and conducts a Monte Carlo experiment. The probabilities from the Monte Carlo analysis should converge to those calculated using the equations in this research. Validation work along these lines has been done by Schulz [51] for the single warhead case.

6.2 Operational Employment

Once the UCAV is fielded, we may want to know the expected number of target kills as the mission progresses. This information is found in Figure 6.4. We can also use this figure when preparing to deploy the UCAV. We have to make a tradeoff between range of the UCAV and number of warheads the UCAV carries (up to a maximum of fourteen due to weight considerations). We could put the max load of warheads on the UCAV, but this would reduce the fuel load and therefore the region being searched. What would be the right combination of fuel and warheads? In this case, let us assume calculations show that with a full complement of warheads ($w = 14$), the UCAV can only cover 40% of the region it could cover if it only carried a payload of two warheads. So we can trade off area covered for max possible number of targets killed (the number of warheads). Let us also

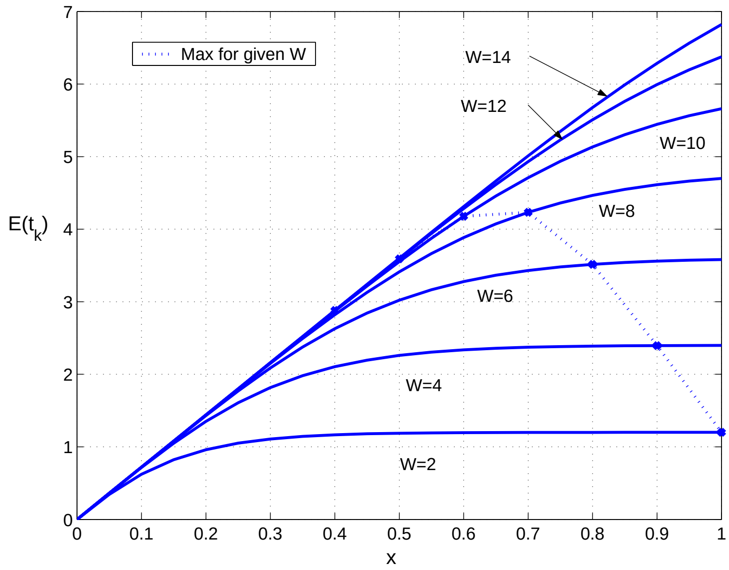

Figure 6.10 Scenario 2: Expected Number of Target Kills and Area Covered Tradeoff, $\lambda_{FT} = 20, \lambda_T = 10, P_k = 0.8, P_{TR} = 0.9, q = 18$

assume that calculations show that starting with two warheads, for every two warheads we add, the UCAV loses 10% of its two warhead range. The decision points are displayed in Figure 6.10. Based on this Figure, we opt to carry ten warheads to maximize the expected number of target kills and expect to kill four targets.

Once the decision is made to reduce the max area covered, we need to either rescale the normalized time (x) or we need to rescale the parameters. We opt to do the latter. To complete the rescaling, we need the old parameters and the x value which will be the new maximum x value (denoted x_m):

$$\lambda_T = \lambda_{T_{old}} x_m = 10(0.6) = 6.0 \; , \tag{6.3}$$

$$\lambda_{FT} = \lambda_{FT_{old}} x_m = 20(0.6) = 12 \; . \tag{6.4}$$

We then re-normalize the x values so that when we have reached the maximum range with $w = 10$ we have $x = 1$. In addition, we will plot the expected value versus $x_c = 1 - x$. The reason for this will become clearer in the next section.

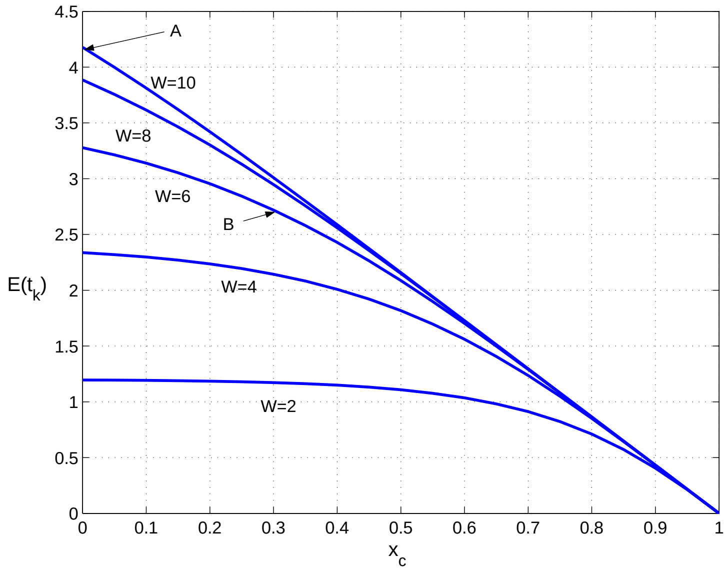

Figure 6.11 Scenario 2: Expected Number of Target Kills with Rescaled Area, $\lambda_{FT} = 12, \lambda_T = 6, P_k = 0.8, P_{TR} = 0.9, q = 18$

6.3 Tactical Decision Making

The UCAV originally had ten warheads and, based on Figure 6.11, we expect four target kills (point A in Figure 6.11). This is based on allowing the UCAV to exhaustively and non-duplicatively search a region until its fuel is exhausted (represented by the normalized time/normalized area covered $x = 1$). However, let us assume that partway into the flight, the UCAV is called out of the battle space to help another UCAV kill some of its targets. It takes a normalized time of $x_c = 0.3$ from the time it leaves its search, assists the other UCAV and gets back to the point it left ($x_c = 0.3$ can be thought of as using 30% of its fuel for this deviation from its original mission). In the process of assisting the other UCAV, this UCAV expended four warheads. Since the amount of fuel available for the search of the original area and the total number of warheads which can be used in the original battle space have both decreased, we now have a new (lower) expected value for the total number of target kills in the original battle space. Figure 6.11 can be used to determine the new expected value (point B). This figure would also be useful for a UCAV that does not leave the battle space, but spends time loitering to improve classification of a target. In this situation, the time spent loitering would be the x_c and the number of warheads would not change from the original.

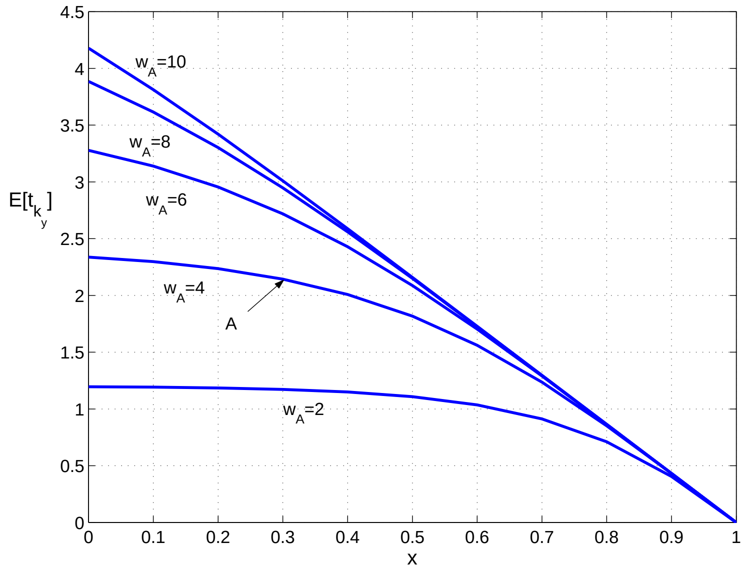

Figure 6.12 Scenario 2: Expected number of Target Kills assuming w_A warheads remain after x, $\lambda_{FT} = 20, \lambda_T = 10, P_k = 0.8, P_{TR} = 0.9, q = 18$

Finally, let us examine a tactical situation where the UCAV started with ten warheads and has covered 30% of the battle space and has expended six warheads. We may then wonder what is the expected number of target kills from this point on. This information can help us decide if we want the UCAV to continue searching or if we want to forsake this originally assigned area to send the UCAV to help another UCAV in its search (or attack). The needed information is found in Figure 6.12, where we see that with four warheads left and 30% of the originally assigned battle space covered, we can expect two more target kills (point A). This would be the cost of assisting another UCAV in its search and attack mission (forsaking the currently assigned area). We note here that Figures 6.11 and 6.12 are equivalent due to the independent and stationary increments inherent in the Poisson process. These figures would not necessarily be equivalent for other scenarios.

If the UCAV was operating autonomously, the UCAV might have an algorithm that continuously updates the probability of at least one more target kill, given its current position. Based on that probability, combined with knowledge of its fuel status, it could decide whether to continue the search or assist another UCAV which has transmitted a possible target location. In this case, the UCAV has covered 30% of its region and has four warheads left. The probability of killing at least one more target is obtained by

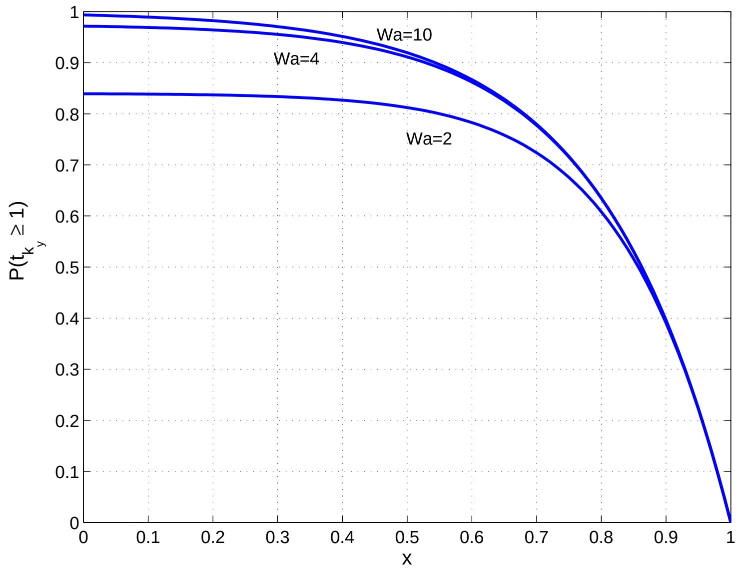

Figure 6.13 Scenario 2: Probability of at least one Target Kill assuming w_A warheads remain after x, $\lambda_{FT} = 20, \lambda_T = 10, P_k = 0.8, P_{TR} = 0.9, q = 18$

$P(\mathcal{K}_{t_k \geq k,x})$. We can see in Figure 6.13 that the probability of at least one more target kill if we continue to search is approximately 97%. Of course, the validity of the results presented in this section are all subject to the validity of the assumed distributions.

VII. *Control Formulation*

In reality, our knowledge of the distribution of targets and false targets is limited. In addition, when we attack an object, we do not know if we attacked a true target or not. We just know the ATR declared the object to be a target. However, if we had a Bomb Damage Assessment (BDA) capability, we could then know (after the fact) that we hit a target. In fact, we could use this knowledge to assist us in making a better guess at the distributions of the targets and false targets. With a better guess/estimate of the distributions, we could improve our chances of attacking targets and avoiding false targets. Our problem could then be viewed as a control type formulation.

In this formulation, we will again focus on Scenario 2. We assume the BDA capability is able to detect whether an attacked object was a target or a false target. The states of the system could be $(\mathcal{T}_{t,x}, \mathcal{F}_{f,x})$ (or $X_{t,f,x}$), λ_T, and λ_{FT}. The actuators would be the UCAV's, the sensors would correspond to the mechanism providing the BDA capability as well as a counter which keeps track of the number of warheads we have used up to this point (or equivalently, the number of warheads left after the last attack).

The question then arises as to the control command. To answer this question, we look to the ROC. Recall that we assumed the ATR was such that a correlation factor was used to decide if the object was a target or a false target. If the correlation was above a certain threshold, the object was declared to be a target. If below the threshold, it was declared to be a false target. This threshold is our candidate control. By way of example, let us assume that we had thought $\lambda_T = 5$, and $\lambda_{FT} = 20$. As we attack objects and conduct BDA, we find that λ_T was higher than expected, say $\lambda_T = 20$. Assuming we can not change to a different sensor with a different ROC curve (whether in reality or in effect via cooperative classification with another UCAV), about the only thing left to us is to change where we are on the ROC curve, via the threshold. In our example, since it looks like we have more targets than we originally thought, we can be more discriminating and raise the threshold thereby lowering P_{TR}. For simplicity, we assume that the threshold parameter, h, and P_{TR} are related as follows;

$$P_{TR} = 1 - h \ . \tag{7.1}$$

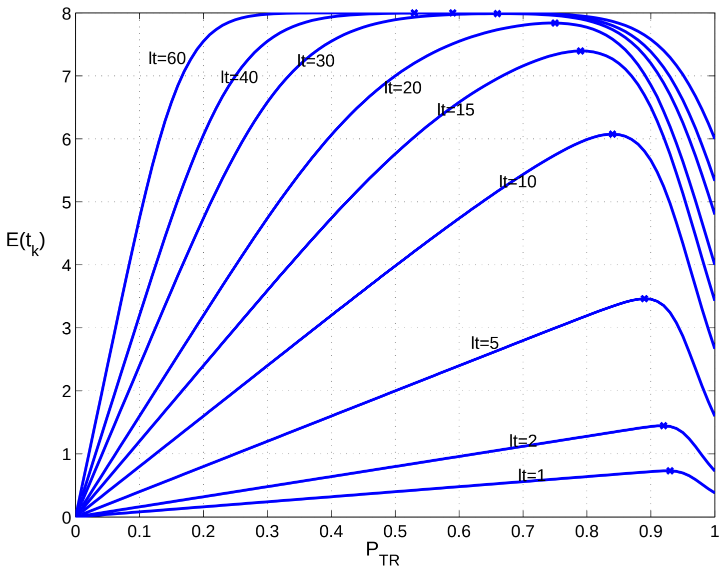

Figure 7.1 Scenario 2: Expected Number of Target Kill as a function of $P_{TR},\ \lambda_T$: $\lambda_{FT} = 20,\ w = 10,\ P_k = 0.8,\ q = 18$

This erroneously assumes that we can know P_{TR} for certain, when in truth we would most likely only know its expected value for certain situations. However, in this chapter as well as Chapter VI, we assume we can know P_{TR}.

By using (7.1) and a ROC curve such as found in Table 6.1 we see that when lowering the probability of correctly classifying a target given we have come across a target (lowering the number of target attacks), we will also lower the number of false target attacks, leaving more warheads available for attacking targets. This, coupled with the fact that we now have more targets than originally thought, will serve to increase the total number of targets attacked. For the rest of this chapter, we seek to maximize the number of target attacks/kills.

Once we have a good estimate of λ_T, we can use that knowledge to increase our expected number of TK's. Figure 7.1 shows this expected number of TA's as a function of P_{TR} and λ_T,. The maximum expected number for each λ_T is marked by a $\times$. Figure 7.2 shows a plot of the best P_{TR} for a given λ_T,(λ_{FT} assumed fixed) to maximize our expected number of target kills. Once we have a good estimate of $\lambda_T,\ \lambda_{FT}$, we can find the P_{TR} (the control) to maximize our expected number of target kills.

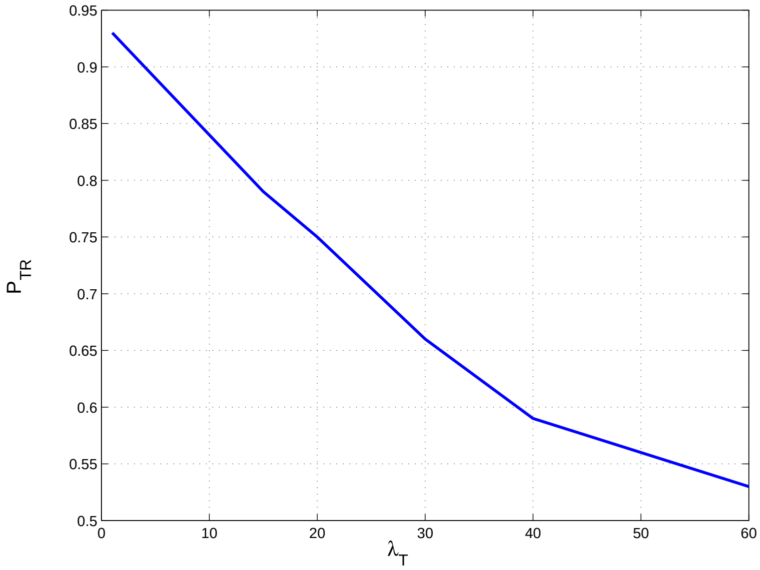

Figure 7.2 Scenario 2: Best P_{TR} for maximum Expected Number of Target Kill as a function of λ_T: $\lambda_{FT} = 20$: $w = 10,\ P_k = 0.8,\ q = 18$

For now, we will guess that along with $\lambda_{FT} = 20$, that $\lambda_T = 5$. We choose the threshold such that $P_{TR} = 0.8$ based on Figures 7.1 and 7.2. Recall that for our sensor model, the ROC curve is defined as

$$P_{FTR} = 1 - P_{TR}^{q} \ .$$

If our BDA was 100% accurate, after each measurement (BDA cycle) we would know the state $X_{t,f,x}$ exactly, but the values for λ_T and λ_{FT} remain uncertain. We need to estimate their correct values. We chose to try and estimate these parameters via hypothesis testing.

A probability which will be very useful in trying to determine the correct values of λ_T and λ_{FT} is the probability of being in state $X_{t,f,x}$ given w_A warheads left after A, $P\left\{X_{t,f,x}|w_A\right\}$. For Scenario 2, this probability is calculated via the following;

$$P\left(X_{t,f,x}|w_A\right) \quad = \quad \frac{P\left\{X_{t,w-w_A-t,x}\right\}}{\sum_{i=0}^{w-w_A} P\left\{X_{i,w-w_A-i,x}\right\}} \ . \tag{7.2}$$

Now we look at two cases $w_A \neq 0$ and $w_A = 0$. For the former case,

$$P\left(X_{t,f,x}|w_A \neq 0\right) = \frac{\frac{(\lambda_{A_T}x)^t}{t!}\frac{(\lambda_{A_{FT}}x)^{w-w_A-t}}{(w-w_A-t)!}e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x}}{\sum_{i=0}^{w-w_A}\frac{(\lambda_{A_T}x)^i}{i!}\frac{(\lambda_{A_{FT}}x)^{w-w_A-i}}{(w-w_A-i)!}e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x}} , \tag{7.3}$$

$$= \frac{\frac{\lambda_{A_T}^t}{t!}\frac{\lambda_{A_{FT}}^{-t}}{(w-w_A-t)!}}{\sum_{i=0}^{w-w_A}\frac{\lambda_{A_T}^i}{i!}\frac{\lambda_{A_{FT}}^{-i}}{(w-w_A-i)!}} , \tag{7.4}$$

$$= \frac{\binom{w-w_A}{t}\left(\frac{\lambda_{A_T}}{\lambda_{A_{FT}}}\right)^t}{\left(1+\frac{\lambda_{A_T}}{\lambda_{A_{FT}}}\right)^{w-w_A}} , \tag{7.5}$$

$$= \frac{\binom{w-w_A}{t}\left(\frac{P_{TR}}{(1-P_{FTR})}\frac{\lambda_T}{\lambda_{FT}}\right)^t}{\left(1+\frac{P_{TR}}{(1-P_{FTR})}\frac{\lambda_T}{\lambda_{FT}}\right)^{w-w_A}} . \tag{7.6}$$

A similar development of the $w_A = 0$ case provides the same answer, so

$$P\left(X_{t,f,x}|w_A\right) = \frac{\binom{w-w_A}{t}\left(\frac{P_{TR}}{(1-P_{FTR})}\frac{\lambda_T}{\lambda_{FT}}\right)^t}{\left(1+\frac{P_{TR}}{(1-P_{FTR})}\frac{\lambda_T}{\lambda_{FT}}\right)^{w-w_A}} . \tag{7.7}$$

We can see that (7.7) does not depend on the values of λ_T and λ_{FT}, only their ratio. However, the expected number of TA or FTA is dependent on their values. We also note that (7.7) does not depend on x. This dependence was cancelled out due to the ratio of λ_{A_T} to $\lambda_{A_{FT}}$. Therefore, any subsequent analysis can be applied to any x. This analysis is obviously only applicable for Scenario 2. Let

$$r \equiv \frac{\lambda_{FT}}{\lambda_T} . \tag{7.8}$$

For now, let us assume that we have four hypotheses on the value of r ($r = 1, 2, 4, 20$). These plots are found in Figure 7.3. Even though the plots in this figure only depend on the ratio of λ_{FT} to λ_T, we must keep in mind that the actual values of λ_{FT} and λ_T affect the expected number of target kills and therefore impact the control value we would use.

We see from this figure that if our measurement told us we had six TA's, then it would be reasonable to assume (out of these four choices) $r = 1$, with potential refinement

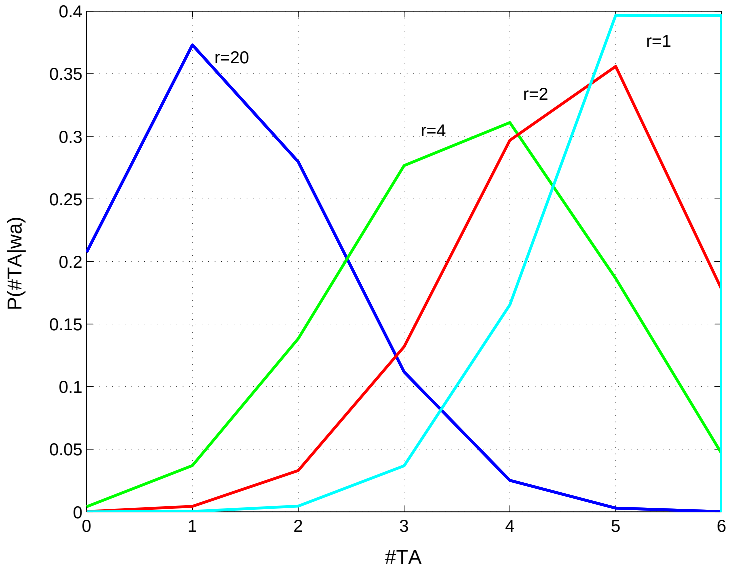

Figure 7.3 Scenario 2: Probability of target attack for various ratios of λ_{FT}/λ_T given four warheads left after A: $P_{TR} = 0.9, q = 18, w = 10$

later as more warheads are dropped. However, having a good estimate of the ratio r is not enough to determine a good control value. We also need a good estimate of λ_T (or λ_{FT}). Once we have good estimates of r and λ_T we can determine λ_{FT} and use plots similar to that found in Figure 7.2 to determine the appropriate control.

We can estimate the correct value of λ_T in a fashion similar to that used in estimating the ratio. In this case, instead of using the statistic $P\{X_{t,f,x}|w_A\}$ as we did for the ratio, we could use the statistic $P\{\mathcal{W}_\mathcal{A} = w_A\}$ as a function of x. Letting $r = \lambda_{FT}/\lambda_T$, (4.85) becomes

$$\begin{aligned} P(w_A : 1 \leq w_A \leq w) &= e^{-(\lambda_{A_T}+\lambda_{A_{FT}})x} \frac{((\lambda_{A_T} + \lambda_{A_{FT}})x)^{w-w_A}}{(w-w_A)!} \\ &= e^{\lambda_T(P_{TR}+r(1-P_{FTR}))x} \frac{(\lambda_T (P_{TR} + r(1-P_{FTR}))\, x)^{w-w_A}}{(w-w_A)!} \qquad (7.9) \end{aligned}$$

We can then plot this probability as a function of x for various λ_T's as in Figure 7.4. If we started with ten warheads and had our sixth attack at $x = 0.28$ and from our previous analysis we had determined that it was most likely that $r = 1$, then we would conclude from Figure 7.4 that of the five choices represented ($\lambda_T = 20, 10, 5, 2, 1$) that $\lambda_T = 20$

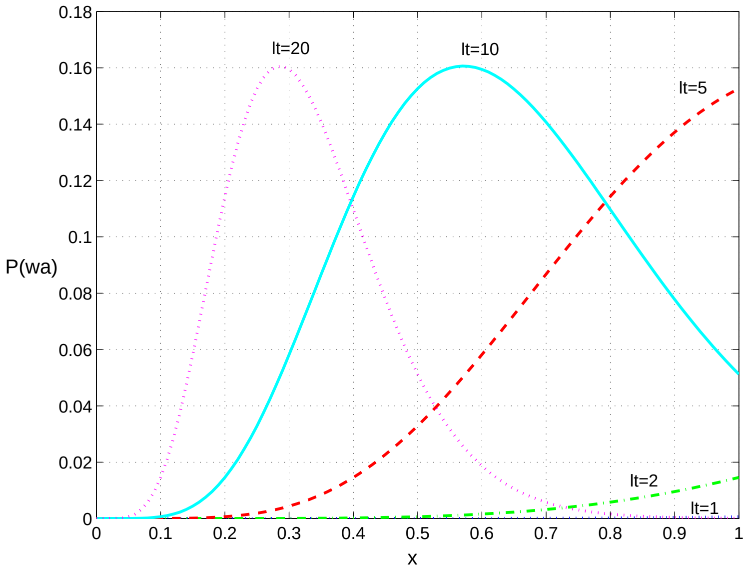

Figure 7.4 Scenario 2: Probability of $w_A = 4$ warheads left after A: $r = 1, P_{TR} = 0.9, q = 18, w = 10$

seems the best choice. Therefore, if $r = 1$ and $\lambda_T = 20$, then $\lambda_{FT} = 20$; and we see from Figure 7.2 that the new command should be the threshold where $P_{TR} = 0.75$. Assuming we have the correct λ_T and r, this new P_{TR} will maximize our number of target kills.

The previous discussion details the control for the estimated parameter issue. Several methods for estimating the parameters λ_T, λ_{FT} could be used along with the previous discussion.

7.1 Hypothesis Testing

Since we do not have just a null and an alternate hypotheses, but instead we have multiple hypotheses, we will need to use a variation of the traditional hypothesis testing. One of the potential methods is the Generalized Sequentially Rejective Bonferroni Test (GSRBT). For an example see [50].

At this point, we note that while Figure 7.2 is to some extent a summary of Figure 7.1, we should not ignore Figure 7.1. We seek the best P_{TR} to maximize the expected number of target kills. But we can see from Figure 7.1 that as λ_T increases (decreasing the ratio r) we get to a point where the maximum number of target kills is unchanging even though the best P_{TR} is changing significantly. This can be very useful information. We can see from Figure 7.1 that we do not have to account for a myriad of hypotheses. For the situation represented by Figure 7.1 ($\lambda_{FT} = 20,\ w = 10,\ \lambda_T$ ranging from 1 to 60, which means r ranges from 20 to 1/3, etc), we can eliminate hypotheses above $\lambda_T = 30$ because the change in expected value is relatively insignificant for a wide range of P_{TR} values. In fact, we decided that $\lambda_T = 20$ was high enough and were able to limit the number of possible hypotheses accordingly. We will limit the complexity by choosing four hypotheses for our analysis of this example. The λ_T's we shall choose are $\lambda_T = 1, 5, 10, 20$ and their corresponding ratios (since $\lambda_{FT} = 20$) are $r = 20, 4, 2, 1$.

For a particular value of λ_T or λ_{FT} there will be a particular ratio (designated r^*) beyond which no significant increase in $E[t_k]$ is observed. In the case represented by Figure 7.1, $\lambda_{FT} = 20$ has $r^* = 1$ (corresponds to $\lambda_T = 20$ in the figure). This r^* value depends on the value for λ_{FT} (or λ_T). Recall that our proposed method is to first estimate r then use that value to estimate λ_T (or λ_{FT}). Therefore, we should not at this point, limit our possible ratios for a particular value of λ_T (or λ_{FT}). Instead we should examine a feasible range of λ_T's (or λ_{FT}'s) and determine the most extreme r^* for that range. This then would be the appropriate limit for the range of our hypotheses. However, since we are merely illustrating a potential use of our research, we will continue with a particular value of λ_{FT}.

Table 7.1 Probabilities of #TA Given $w_A = 4$ and Given Several Hypotheses

#TA	H20 $r = 20$	H4 $r = 4$	H2 $r = 2$	H1 $r = 1$
0	0.2074	0.0041	0.0002	0.0000
1	0.3730	0.0369	0.0044	0.0003
2	0.2796	0.1384	0.0330	0.0046
3	0.1118	0.2766	0.1320	0.0368
4	0.0251	0.3110	0.2967	0.1654
5	0.0030	0.1865	0.3558	0.3966
6	0.0002	0.0466	0.1778	0.3964
p value	0.0002	0.0466	0.1778	0.3964

The first step in the GSRBT process is to make an observation and then find the p values of each hypothesis. Let us assume that BDA tells us that we have hit six actual targets. In that case the p values for our hypotheses are found on the bottom row of Table 7.1. Let us also assume that we had decided *a priori* to set α (the probability of a Type I error) of $\alpha = 0.05$ (a typical value).

We can incorporate any *a priori* information on the hypotheses by using a set of positive real constants $c_1, \ldots, c_P$, which have values directly proportional to the importance of the individual hypotheses. We then define new p values defined as $S_i = p_i/c_i$. In our case we will assume we have no *a priori* information and so will set all the constants equal to one.

We order the new S_i values in ascending order, $S_{(1)} \leq S_{(2)} \leq \cdots \leq S_{(P)}$, letting $c_{(i)}$ and $H_{(i)}$ be the corresponding constants and hypotheses respectively. In addition, we define $\alpha_i = \alpha / \sum_{j=i}^{P} c_{(j)}$. Sequential tests can now be conducted as follows: As i goes from 0 to P, if $S_{(i)} \leq \alpha_i$ we reject $H_{(i)}$ and continue on to test the next increment. If $S_{(j)} > \alpha_j$ then we fail to reject $H_{(j)}$ $(j = i, \ldots, P)$.

In our example, with an observation of six TA, we see that the hypotheses are already in ascending order. That is, $H_{(1)} = H20$, $H_{(2)} = H4$, $H_{(3)} = H2$, $H_{(4)} = H1$. Our first test in the sequential testing is for $H_{(1)}$. We see that $S_{(1)} = 0.0002 \leq \alpha_1 = 0.05/4$, therefore $H_{(1)}$ is rejected. However, since $S_{(1)} = 0.0466 > \alpha_2 = 0.05/3$ which means we fail to reject $H_{(i)}$ $(i = 2, 3, 4)$.

The reason for the division of α by the number of tests (and then by decreasing amounts thereafter) is to correct for a problem that occurs when performing k multiple independent significance tests each at the α level. The probability of incorrectly rejecting the null (Type I error) at least once is $1 - (1 - \alpha)^k$. For example, with $k = 4$ and $\alpha = 0.05$, there is a 19% chance of at least one of the four tests being declared significant when the null hypothesis is true. However, it is not certain that the condition of performing four multiple independent tests is met. If this requirement could be relaxed then we would be able to reject $H_{(2)}$ as well. In addition, there are other less restrictive modifications to account for this error.

An alternative approach to this problem is provided by maximum likelihood methods. In general and simplified terms, these methods would multiply the resulting command from each hypothesis by the probability that that hypothesis is true. The resulting command would be a weighted sum of each hypothesis' command with the heavier weight placed on the most probable hypothesis. This might be more fruitful especially when we note that the real range of possible r's is not limited to just these four, but could be any real number on the interval.

VIII. Cooperation

This chapter examines uses of the probability factors for cooperative behavior. We will examine some very basic cooperative schemes for two UCAV's. For mathematical tractability we will concentrate on Scenario 2. In all schemes we will examine two rules of engagement (ROE). The schemes, ROE's, and resulting confusion matrix parameters are similar to those devised by Jacques [28] and Jacques and Pachter [29]. The first ROE is that both vehicles must agree that an object is a target before attacking. The second ROE is that at least one must declare the object a target before attacking. We do not address which UCAV attacks the target.

8.1 Rules of Engagement

As far as the sensors are concerned, by using these two ROE's, we have, in effect, combined both sensors to produce a meta-sensor. This meta-sensor's properties are determined by the individual UCAV's sensors and the ROE that governs the classification process. Let us assume that both UCAV's have the same type of sensors such that their sensor parameters are the same. Further, let us assume for the sake of simplicity, that the declaration of an object as a target by one sensor is independent of the declaration of that object by the other sensor. If we are working under ROE 1 the meta-sensor's confusion matrix is as shown in Table 8.1. We see that the sensor parameters for ROE 1 (both must

Table 8.1 Meta-Sensor Confusion Matrix for ROE 1

		Object Encountered	
Object Reported as		T	FT
	T	P_{TR}^2	$(1 - P_{FTR})^2$
	FT	$1 - P_{TR}^2$	$1 - (1 - P_{FTR})^2$

declare it a target) is

$$P_{TR_B} = P_{TR}^2 \, , \tag{8.1}$$

$$P_{FTR_B} = 1 - (1 - P_{FTR})^2 \, . \tag{8.2}$$

The meta-sensor's P_{TR} (i.e. P_{TR_B}) is lower than the single sensor's, but the meta-sensor's P_{FTR} (i.e. P_{FTR_B}) is higher than the single sensor's. Therefore, we will attack targets less often, but we will also attack false targets less often. Note this is true even if our P_{TR} and P_{FTR} could be altered independently (i.e. not related via the ROC).

If working under ROE 2 the confusion matrix is found in Table 8.2. From Table 8.2,

Table 8.2 Meta-Sensor Confusion Matrix for ROE 2

		Object Encountered	
Object		T	FT
Reported	T	$1-(1-P_{TR})^2$	$1-P_{FTR}^2$
as	FT	$(1-P_{TR})^2$	P_{FTR}^2

we see that the sensor parameters for ROE 2 (either must declare it a target) is

$$P_{TR_E} = 1-(1-P_{TR})^2 \ , \tag{8.3}$$

$$P_{FTR_E} = P_{FTR}^2 \ . \tag{8.4}$$

So then, the meta-sensor's P_{TR} (i.e. P_{TR_E}) is higher than the single sensor's, but the meta-sensor's P_{FTR} (i.e. P_{FTR_E}) is lower than the single sensor's. We will be more likely to have target attacks, but also more likely to have false target attacks (given everything else is the same). Again, this is independent of whether a ROC curve is governing P_{TR} and P_{FTR} or not. With this information we may consider various cooperative schemes.

8.2 Scheme 1: Travel the Same Path

The first cooperative scheme will be where both UCAV's travel the same path together. Figure 8.1 depicts this scheme

In this scheme, each UCAV sees everything the other one sees and both have the same probability of coming across a T or FT. And since the two UCAV's are flying together, either UCAV's warheads are capable of hitting any target.

Since either UCAV's warheads can be used and since we are utilizing both UCAV's sensors simultaneously on the same field, we in effect have a super-UCAV. One in which

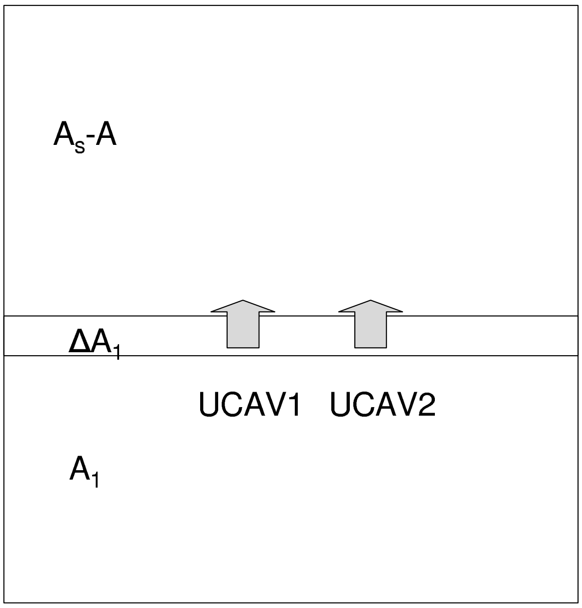

Figure 8.1 Cooperative Scheme 1

the super-UCAV sensor is defined by the ROE's described in Tables 8.1 and 8.2 and who has $w_B = w_{ucav1} + w_{ucav2}$ warheads.

For Scenario 2 we will need to define the following variables for the Poisson distributions.

$$\lambda_{A_{T_B}} = P_{TR_B}\lambda_T \ , \tag{8.5}$$

$$\lambda_{A_{FT_B}} = (1 - P_{FTR_B})\lambda_{FT} \ . \tag{8.6}$$

This, then, makes the Scenario 2 equations as follows

$$P\left(X_{t,f,x} : t + f < w_B\right) = \frac{\left(\lambda_{A_{T_B}} x\right)^t}{t!} \frac{\left(\lambda_{A_{FT_B}} x\right)^f}{f!} e^{-\left(\lambda_{A_{T_B}} + \lambda_{A_{FT_B}}\right)x} \ , \tag{8.7}$$

$$P\left(X_{t,f,x} : t + f = w_B\right) = \lambda^t_{A_{T_B}} \lambda^f_{A_{FT_B}} \left(\frac{w_B}{t!f!}\right) \frac{\gamma(w_B, \left(\lambda_{A_{T_B}} + \lambda_{A_{FT_B}}\right) x)}{\left(\lambda_{A_{T_B}} + \lambda_{A_{FT_B}}\right)^{w_B}} \ . \tag{8.8}$$

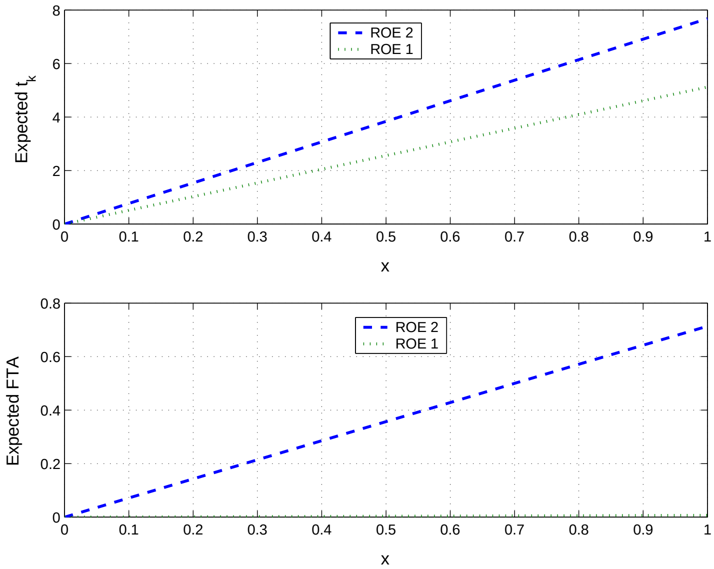

Figure 8.2 Cooperative Scheme 1 Comparison of ROE 1 and 2: $w_{ucav1} = 10$, $\lambda_T = 10$, $\lambda_{FT} = 20, P_{TR} = 0.8$, $q = 18$

Similarly, for ROE 2 we have

$$\lambda_{A_{T_E}} = P_{TR_E}\lambda_T \ , \tag{8.9}$$

$$\lambda_{A_{FT_E}} = (1 - P_{FTR_E})\lambda_{FT} \ , \tag{8.10}$$

and

$$P\left(X_{t,f,x} : t+f < w_B\right) = \frac{\left(\lambda_{A_{T_E}} x\right)^t}{t!} \frac{\left(\lambda_{A_{FT_E}} x\right)^f}{f!} e^{-\left(\lambda_{A_{T_E}} + \lambda_{A_{FT_E}}\right)x} \ , \tag{8.11}$$

$$P\left(X_{t,f,x} : t+f = w_B\right) = \lambda_{A_{T_E}}^t \lambda_{A_{FT_E}}^f \left(\frac{w_B}{t!f!}\right) \frac{\gamma(w_B, \left(\lambda_{A_{T_E}} + \lambda_{A_{FT_E}}\right) x)}{\left(\lambda_{A_{T_E}} + \lambda_{A_{FT_E}}\right)^{w_B}} \ . \tag{8.12}$$

Now we can compare the two ROE's. To do this we will plot the expected number of target kills vs x (percentage into the battle space) and expected number of false target attacks vs x. We see this comparison in Figure 8.2. Note what happens when we make the density of the targets larger than the density of the false targets as seen in Figure 8.3. In this figure we see a plateau effect, in this case, due to the number of warheads.

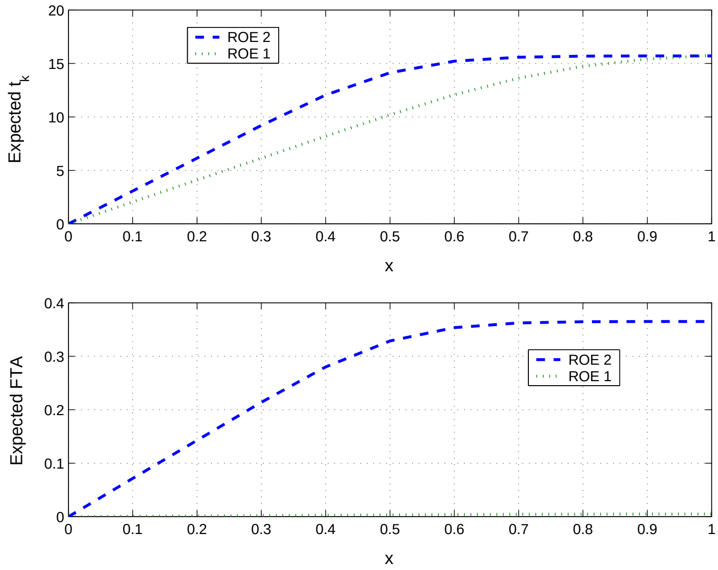

Figure 8.3 Cooperative Scheme 1 Comparison of ROE 1 and 2: $w_{ucav1} = 10,\ \lambda_T = 40,\ \lambda_{FT} = 20,\ P_{TR} = 0.8,\ q = 18,\ P_k = 0.8$

We see from these figures that ROE 2 is the best if we are willing to accept the resulting number of false target attacks. However, if we are not willing to accept the requisite number of false target attacks, then ROE 1 would be the ROE to choose.

Figure 8.4 shows the same info as a function of P_{TR}. From this figure we can evaluate the ROE's for any P_{TR}. In this example, we see from Figure 8.4 that ROE 1 is more advantageous for both the expected number of target kills as well as the expected number of FTA's when $P_{TR} > .8$. For $P_{TR} < .7$ ROE 2 is more advantageous for expected number of target kills with approximately the same expected number of FTA's. This kind of information is useful in determining the desired ROE for a given situation.

8.3 Scheme 2: Travel Parallel Paths

For Scheme 2, the two UCAV's follow parallel paths but with the capability to instantaneously look at the other path to help in the classification when directed by the UCAV on that path. The UCAVs can also drop warheads on either path. This either involves invoking a simplifying assumption of instantaneous transport from one path to the other, or maybe a bit more realistic, the two UCAV's flying side by side but using a

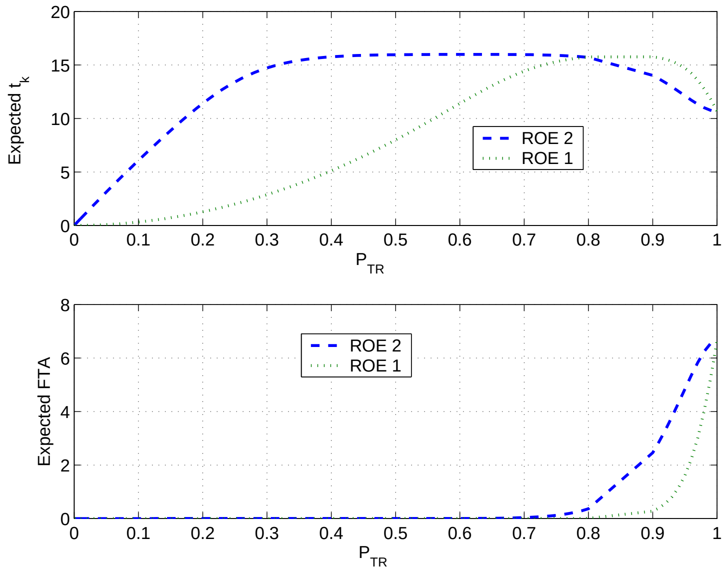

Figure 8.4 Cooperative Scheme 1 Comparison of ROE 1 and 2 and P_{TR}: $w_{ucav1} = 10, \lambda_T = 40, \lambda_{FT} = 20, q = 18, P_k = 0.8$

sensor looking out the side of the UCAV. Each UCAV's sensor concentrates their search on their path but can instantly swing the sensors field of view to the other path when requested by the other UCAV. Figure 8.5 depicts this scheme.

With this scenario, we will assume that when UCAV1 sees an object, it asks UCAV2 to confirm regardless of if UCAV1 classified it as a target or a false target.

The parameters P_{TR_E}, P_{FTR_E}, P_{TR_B}, P_{FTR_B} are the same as in the previous scheme, but the distributions of the targets and false targets are now different. We will designate the distribution of the targets and false targets on UCAV1's side as λ_{T_1}, λ_{FT_1}, respectively. Similarly, on UCAV2's side we have λ_{T_2}, $\lambda_{A_{FT_2}}$.

Therefore,

$$\lambda_{A_{T_{E1}}} = P_{TR_E}\lambda_{T_1} \ , \tag{8.13}$$

$$\lambda_{A_{FT_{E1}}} = (1 - P_{FTR_E})\lambda_{A_{FT_1}} \ , \tag{8.14}$$

$$\lambda_{A_{T_{B1}}} = P_{TR_B}\lambda_{T_1} \ , \tag{8.15}$$

$$\lambda_{A_{FT_{B1}}} = (1 - P_{FTR_B})\lambda_{A_{FT_1}} \ , \tag{8.16}$$

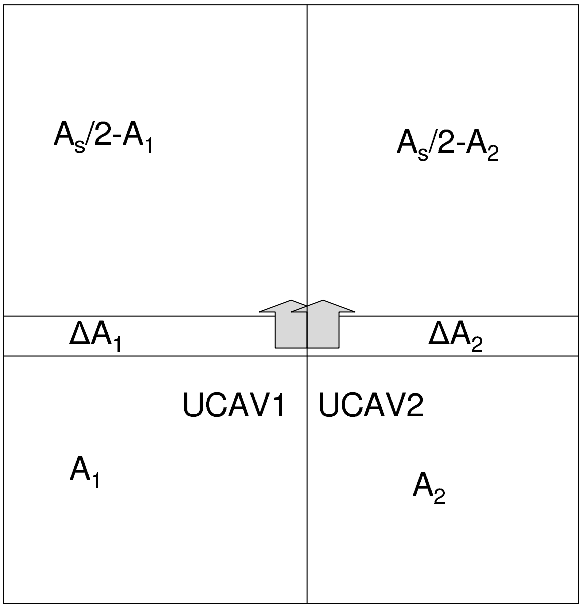

Figure 8.5 Cooperative Scheme 2

and similar equations for UCAV2's territory.

As a side note, if we were to take the same battle space we had in Scheme 1 and use cooperative Scheme 2 on it, we would have

$$\lambda_{T_1} = \lambda_{T_2} = \frac{\lambda_T}{2} \ , \tag{8.17}$$

$$\lambda_{FT_1} = \lambda_{FT_2} = \frac{\lambda_{FT}}{2} \ . \tag{8.18}$$

In addition, we must take into account the various combinations of TA's and FTA's in each of the areas when finding the probability that the system has attacked a certain number of targets and false targets.

$$P_{t,f,system}^{(t+f<w_B)} = \sum_{t_1=0}^{t} \sum_{f_1=0}^{f} P_{t_1,f_1,U1}^{(t+f<w_B)} P_{t-t_1,f-f_1,U2}^{(t+f<w_B)} \ . \tag{8.19}$$

We find

$$P^{(t+f<w_B)}_{t,f,system} = \sum_{t_1=0}^{t}\sum_{f_1=0}^{f} e^{-(\lambda_{A_{T_{B1}}}+\lambda_{A_{FT_{B1}}}+\lambda_{A_{T_{B2}}}+\lambda_{A_{FT_{B2}}})x}\frac{(\lambda_{A_{T_{B1}}})^{t_1}}{t_1!} \times \frac{(\lambda_{A_{FT_{B1}}})^{f_1}}{f_1!}\frac{(\lambda_{A_{T_{B2}}})^{t-t_1}}{(t-t_1)!}\frac{(\lambda_{A_{FT_{B2}}})^{f-f_1}}{(f-f_1)!}\ , \tag{8.20}$$

$$P^{(t+f=w_B)}_{t,f,system} = \sum_{t_1=0}^{t}\sum_{f_1=0}^{f} C_{t_1,f_1,t-t_1,f-f_1} \times \frac{\gamma(w_B,(\lambda_{A_{T_{B1}}}+\lambda_{A_{FT_{B1}}}+\lambda_{A_{T_{B2}}}+\lambda_{A_{FT_{B2}}})x)}{(\lambda_{A_{T_{B1}}}+\lambda_{A_{FT_{B1}}}+\lambda_{A_{T_{B2}}}+\lambda_{A_{FT_{B2}}})^{w_B}}\ , \tag{8.21}$$

where each subscript of C has a term associated with it and when a subscript is zero, the corresponding term is zero as well,

$$\begin{aligned} C_{t_1,f_1,t-t_1,f-f_1} = & \frac{(\lambda_{A_{T_{B1}}})^{t_1}}{(t_1-1)!}\frac{(\lambda_{A_{FT_{B1}}})^{f_1}}{f_1!}\frac{(\lambda_{A_{T_{B2}}})^{t-t_1}}{(t-t_1)!}\frac{(\lambda_{A_{FT_{B2}}})^{f-f_1}}{(f-f_1)!}+ \\ & \frac{(\lambda_{A_{T_{B1}}})^{t_1}}{t_1!}\frac{(\lambda_{A_{FT_{B1}}})^{f_1}}{(f_1-1)!}\frac{(\lambda_{A_{T_{B2}}})^{t-t_1}}{(t-t_1)!}\frac{(\lambda_{A_{FT_{B2}}})^{f-f_1}}{(f-f_1)!}+ \\ & \frac{(\lambda_{A_{T_{B1}}})^{t_1}}{t_1!}\frac{(\lambda_{A_{FT_{B1}}})^{f_1}}{f_1!}\frac{(\lambda_{A_{T_{B2}}})^{t-t_1}}{(t-t_1-1)!}\frac{(\lambda_{A_{FT_{B2}}})^{f-f_1}}{(f-f_1)!}+ \\ & \frac{(\lambda_{A_{T_{B1}}})^{t_1}}{t_1!}\frac{(\lambda_{A_{FT_{B1}}})^{f_1}}{f_1!}\frac{(\lambda_{A_{T_{B2}}})^{t-t_1}}{(t-t_1)!}\frac{(\lambda_{A_{FT_{B2}}})^{f-f_1}}{(f-f_1-1)!}\ . \end{aligned} \tag{8.22}$$

We can simplify further, then (8.20) becomes

$$P^{(t+f<w_B)}_{t,f,system} = \sum_{t_1=0}^{t}\sum_{f_1}^{f} e^{-(\lambda_{A_{T_{B1}}}+\lambda_{A_{FT_{B1}}}+\lambda_{A_{T_{B2}}}+\lambda_{A_{FT_{B2}}})x}\frac{1}{t_1!(t-t_1)!}\frac{1}{f_1!(f-f_1)!} \times \left(\lambda_{A_{T_{B1}}}x\right)^{t_1}\left(\lambda_{A_{T_{B2}}}x\right)^{t-t_1}\left(\lambda_{A_{FT_{B1}}}x\right)^{f_1}\left(\lambda_{A_{FT_{B2}}}x\right)^{f-f_1}\ , \tag{8.23}$$

$$
\begin{aligned}
P_{t,f,system}^{(t+f<w_B)} = & \sum_{t_1=0}^{t}\sum_{f_1}^{f} e^{-(\lambda_{A_{T_{B1}}}+\lambda_{A_{FT_{B1}}}+\lambda_{A_{T_{B2}}}+\lambda_{A_{FT_{B2}}})x}\frac{t!}{t_1!(t-t_1)!}\frac{f!}{f_1!(f-f_1)!} \\
& \times\left(\frac{\lambda_{A_{T_{B1}}}x}{\lambda_{A_{T_{B2}}}x}\right)^{t_1}\frac{\left(\lambda_{A_{T_{B2}}}x\right)^{t}}{t!}\left(\frac{\lambda_{A_{FT_{B1}}}x}{\lambda_{A_{FT_{B2}}}x}\right)^{f_1}\frac{\left(\lambda_{A_{FT_{B2}}}x\right)^{f}}{f!}\ ,
\end{aligned}
\tag{8.24}
$$

$$
\begin{aligned}
= & \quad e^{-(\lambda_{A_{T_{B1}}}+\lambda_{A_{FT_{B1}}}+\lambda_{A_{T_{B2}}}+\lambda_{A_{FT_{B2}}})x}\frac{(\lambda_{A_{T_{B2}}}x)^t}{t!}\frac{(\lambda_{A_{FT_{B2}}}x)^f}{f!} \\
& \times\sum_{t_1=0}^{t}\binom{t}{t_1}\left(\frac{\lambda_{A_{T_{B1}}}}{\lambda_{A_{T_{B2}}}}\right)^{t_1}(1)^{t-t_1}\sum_{f_1=0}^{f}\binom{f}{f_1}\left(\frac{\lambda_{A_{FT_{B1}}}}{\lambda_{A_{FT_{B2}}}}\right)^{f_1}(1)^{f-f_1}\ ,
\end{aligned}
\tag{8.25}
$$

$$
\begin{aligned}
P_{t,f,system}^{(t+f<w_B)} = & \ e^{-(\lambda_{A_{T_{B1}}}+\lambda_{A_{FT_{B1}}}+\lambda_{A_{T_{B2}}}+\lambda_{A_{FT_{B2}}})x}\frac{(\lambda_{A_{T_{B2}}}x)^t}{t!}\frac{(\lambda_{A_{FT_{B2}}}x)^f}{f!} \\
& \times\left(1+\frac{\lambda_{A_{T_{B1}}}}{\lambda_{A_{T_{B2}}}}\right)^{t}\left(1+\frac{\lambda_{A_{FT_{B1}}}}{\lambda_{A_{FT_{B2}}}}\right)^{f}\ .
\end{aligned}
\tag{8.26}
$$

To simplify (8.21) requires a simplification of the summation of the C term.

$$
\begin{aligned}
\sum_{t_1=0}^{t}\sum_{f_1=0}^{f} C_{t_1,f_1,t-t_1,f-f_1} = & \\
\left(\lambda_{A_{T_{B2}}}\right)^{t}\left(\lambda_{A_{FT_{B2}}}\right)^{f}\sum_{t_1=0}^{t}\sum_{f_1=0}^{f}\left(\frac{\lambda_{A_{T_{B1}}}}{\lambda_{A_{T_{B2}}}}\right)^{t_1}\left(\frac{\lambda_{A_{FT_{B1}}}}{\lambda_{A_{FT_{B2}}}}\right)^{f_1} & \\
\times\Bigg(\frac{1}{(t_1-1)!(t-t_1)!}\frac{1}{f_1!(f-f_1)!}+\frac{1}{(t_1)!(t-t_1)!}\frac{1}{(f_1-1)!(f-f_1)!}+ & \\
\frac{1}{(t_1)!(t-t_1-1)!}\frac{1}{f_1!(f-f_1)!}+\frac{1}{(t_1)!(t-t_1)!}\frac{1}{f_1!(f-f_1-1)!}\Bigg)\ , &
\end{aligned}
\tag{8.27}
$$

$$
\begin{aligned}
= & \ \left(\lambda_{A_{T_{B2}}}\right)^{t}\left(\lambda_{A_{FT_{B2}}}\right)^{f}\sum_{t_1=0}^{t}\sum_{f_1=0}^{f}\frac{1}{(t_1)!(t-t_1)!}\left(\frac{\lambda_{A_{T_{B1}}}}{\lambda_{A_{T_{B2}}}}\right)^{t_1} \\
& \times\frac{1}{f_1!(f-f_1)!}\left(\frac{\lambda_{A_{FT_{B1}}}}{\lambda_{A_{FT_{B2}}}}\right)^{f_1}(t+f)\ ,
\end{aligned}
\tag{8.28}
$$

$$
\begin{aligned}
&= \frac{t+f}{t!f!}\left(\lambda_{A_{T_{B2}}}\right)^t\left(\lambda_{A_{FT_{B2}}}\right)^f\sum_{t_1=0}^{t}\binom{t}{t_1}\left(\frac{\lambda_{A_{T_{B1}}}}{\lambda_{A_{T_{B2}}}}\right)^{t_1}(1)^{t-t_1} \\
&\qquad \times\sum_{f_1=0}^{f}\binom{f}{f_1}\left(\frac{\lambda_{A_{FT_{B1}}}}{\lambda_{A_{FT_{B2}}}}\right)^{f_1}(1)^{f-f_1} \ ,
\end{aligned} \tag{8.29}
$$

$$
\begin{aligned}
&= \frac{(t+f)(t+f-1)!}{t!f!}\frac{1}{\Gamma(t+f)}\left(\lambda_{A_{T_{B2}}}\right)^t\left(1+\frac{\lambda_{A_{T_{B1}}}}{\lambda_{A_{T_{B2}}}}\right)^t \\
&\qquad \times\left(\lambda_{A_{FT_{B2}}}\right)^f\left(1+\frac{\lambda_{A_{FT_{B1}}}}{\lambda_{A_{FT_{B2}}}}\right)^f \ .
\end{aligned} \tag{8.30}
$$

Therefore, we have

$$
\begin{aligned}
P_{t,f,system}^{(t+f=w_B)} &= \binom{w_B}{t}\lambda_{A_{T_{B2}}}^t\lambda_{A_{FT_{B2}}}^f\left(1+\frac{\lambda_{A_{T_{B1}}}}{\lambda_{A_{T_{B2}}}}\right)^t\left(1+\frac{\lambda_{A_{FT_{B1}}}}{\lambda_{A_{FT_{B2}}}}\right)^f \\
&\quad\times\frac{\gamma(w_B,(\lambda_{A_{T_{B1}}}+\lambda_{A_{FT_{B1}}}+\lambda_{A_{T_{B2}}}+\lambda_{A_{FT_{B2}}})x)}{\Gamma(w_B)(\lambda_{A_{T_{B1}}}+\lambda_{A_{FT_{B1}}}+\lambda_{A_{T_{B2}}}+\lambda_{A_{FT_{B2}}})^{w_B}} \ .
\end{aligned} \tag{8.31}
$$

We note what happens when the two areas (UCAV1 side and UCAV2 side) have the same distribution. In fact we will incorporate (8.17) and (8.18). We have

$$
\lambda_{A_{T_{B1}}} = \lambda_{A_{T_{B2}}} = \frac{\lambda_{A_{T_B}}}{2} \ , \tag{8.32}
$$

$$
\lambda_{A_{FT_{B1}}} = \lambda_{A_{FT_{B2}}} = \frac{\lambda_{A_{FT_B}}}{2} \ . \tag{8.33}
$$

Equation (8.26) becomes

$$
P_{t,f,system}^{(t+f<w_B)} = e^{-\left(\lambda_{A_{T_B}}+\lambda_{A_{FT_B}}\right)x}\frac{\left(\frac{\lambda_{A_{T_B}}}{2}x\right)^t}{t!}\frac{\left(\frac{\lambda_{A_{FT_B}}}{2}x\right)^f}{f!}(2)^t(2)^f \ , \tag{8.34}
$$

$$
= e^{-\left(\lambda_{A_{T_B}}+\lambda_{A_{FT_B}}\right)x}\frac{\left(\lambda_{A_{T_B}}x\right)^t}{t!}\frac{\left(\lambda_{A_{FT_B}}x\right)^f}{f!} \ , \tag{8.35}
$$

which is the same as in Scheme 1 (as would be expected). We also see that (8.31) becomes

$$\begin{aligned} P_{t,f,system}^{(t+f=w_B)} = & \binom{w_B}{t} \left(\frac{\lambda_{A_{T_B}}}{2}\right)^t \left(\frac{\lambda_{A_{FT_B}}}{2}\right)^f (2)^t (2)^f \\ & \times \frac{\gamma(w_B, \left(\lambda_{A_{T_B}} + \lambda_{A_{FT_B}}\right) x)}{\Gamma(w_B) \left(\lambda_{A_{T_B}} + \lambda_{A_{FT_B}}\right)^{w_B}} , \end{aligned} \tag{8.36}$$

$$P_{t,f,system}^{(t+f=w_B)} = \binom{w_B}{t} \lambda_{A_{T_B}}^t \lambda_{A_{FT_B}}^{w_B-t} \frac{\gamma(w_B, \left(\lambda_{A_{T_B}} + \lambda_{A_{FT_B}}\right) x)}{\Gamma(w_B) \left(\lambda_{A_{T_B}} + \lambda_{A_{FT_B}}\right)^{w_B}} , \tag{8.37}$$

which is, again, the same as in Scheme 1.

IX. Conclusions

9.1 Summary

Various probabilities of a multi-warhead UCAV searching an area consisting of targets and false targets were presented. The process was modelled using two methods. In the first method, an event was described as a series of attacks. The probability of the occurrence of that event was then obtained by computing the probability that the final attack which defines that event occurs in $[x, x + dx]$ and integrating that probability as x goes from 0 to 1. This gives the probability of occurrence of the event over the battle space. In the second method, the process is modelled as a Markov chain and the Chapman-Kolmogorov equations for the probabilities of the states of the system are developed. These probabilities are then combined in various ways to compute the probabilities of the event in question. Using either method, we were able to provide expressions for the probabilities of key events and expected values, regardless of assumed distributions. We then evaluated several probabilities and expected values for specific distributions associated with six scenarios.

Scenario 1: A single target uniformly distributed throughout $\mathsf{A_s}$ and a Poisson field of false targets.

Scenario 2: A Poisson field of targets and a Poisson field of false targets.

Scenario 3: N targets uniformly distributed, and a Poisson field of false targets.

Scenario 4: N uniformly distributed targets, and M uniformly distributed false targets.

Scenario 5: N targets distributed according to a circular normal distribution centered at the origin and a Poisson field of false targets.

Scenario 6: N targets distributed according to a circular normal distribution and M false targets distributed according to a circular normal distribution.

Examples were provided in which the results of this research can be used as decision factors for either the design and/or operation of a multi-munition UCAV in a search and attack mode. At a minimal level, probabilities of target attack and mission success can be analytically determined. In addition, though not included in this research, the expected life spans of munition and target can be computed. Furthermore, the examples showed how these calculated probabilities could be utilized to make acquisition, operational as well as tactical decisions. Acquisition decisions may come in the form of determining cost effectiveness and trade-off studies such as deciding whether to spend scarce resources improving sensors on the UCAVs (i.e. changing the values of P_{TR} and/or P_{FTR}), increasing warhead effectiveness (i.e. increasing P_k), or buying more UCAVs / increasing each UCAV warhead capacity (i.e. increasing w). Design considerations could utilize the expected life spans of munition and target when making any decisions regarding max flight time of the munition. Operational decisions could be made in terms of deciding the number of warheads to place on the UCAV when sending it to a battle space given a probable number and/or layout of targets or false targets. The expected number of target kills and the expected life spans of munition and target could then be the design factors in these mission planning / resource allocation decisions. Tactical decisions could take place within the UCAVs themselves or the operators of the UCAV could make the decisions. For example, $E[\mathcal{K}_{t_{k_y}}|\mathcal{W}_{\mathcal{A}} = w_A]$ or $P(\mathcal{K}_{t_{k_y} \geq k,y})$ can be used as real time decision factors in online algorithms to determine if the remaining time should be spent searching for another target or attacking (or re-attacking) a previously designated target. The problem was also formulated as a control problem. Control inputs were defined for estimated target and false target distribution parameters. A particular probability was proposed for use in the estimation process. A cursory investigation was conducted on using that probability and hypotheses testing as a means to estimate those parameters. Another potential method which may be more suited to the problem is maximum likelihood estimation using the proposed probability.

In addition, some rules of engagement for several cooperation schemes were examined. The long term goal is to develop an analytic tool to reliably assess the benefits of autonomous vs. cooperative operations. Such a tool should prove to be very useful to weapon system designers.

9.2 Contributions

The contributions of this dissertation are as follows:

1. The derivation of analytic expressions of relevant probabilities for a multiple warhead UCAV in Scenarios 1 thru 6 as defined in Chapter II. Besides providing analytic rigor to the field, this work is filling a void in search theory in the area of continuous search for stationary targets among multiple false targets using a multi-warhead UCAV.

2. Illustration of possible applications of this research to design of UCAV systems, as well as their operational and tactical employment.

3. Illustration of a method for evaluation of Rules of Engagement (ROE's) for cooperative schemes for multi-warhead UCAV's. In this method, the effect of the cooperative scheme on the distribution parameters was determined. Then the effect of the ROE on the confusion matrix was computed. Then a parametric analysis can be performed with which we can compare the effect of the ROE's and cooperative schemes on performance, e.g. the expected number of target kills.

4. Introduction of the idea of estimating the distribution parameters and formulating the problem as a control problem. A particular probability was proposed for use in the estimation process. This probability should be useful whatever the estimation scheme, although we performed a cursory look at a particular estimation scheme.

9.3 Recommendations for Further Research

1. Pursue control formulation. Conduct more study on possible estimation methods to include maximum likelihood methods. Hypotheses testing was only given a cursory look. This should be examined further as well as maximum likelihood methods and other estimation methods.

2. Incorporate into the analytics the concept that P_{TR} is a random variable. This dissertation made the simplifying assumption that P_{TR} was deterministic. This could be a matter of conditioning the probabilities on another factor; that P_{TR} is a certain value. We would then multiply the conditional probability by the probability of that P_{TR} value and integrate over the possible P_{TR} values. This would of course require assuming a distribution of the P_{TR} value.

3. Pursue several more cooperative schemes, compare the two ROE's for those schemes and compare schemes. First try opposing paths; recognizing that once a UCAV covers an area, the probability distributions for the targets and false targets have changed from given *a priori* distributions to different *a posteriori* distributions. Also look at a cooperative scheme where the UCAV's have orthogonal paths. Then start to look at three or more UCAV's cooperating.

4. Incorporate variances into the analysis. We used expected values in our applications. However, a user does not want to know they have a good system given enough missions, they want to know their particular mission has a good chance of success.

5. Incorporate multiple types of targets. This then could mean new ROE's. For example; attack priority 1 targets immediately (without confirmation), attack priority 2 targets upon confirmation of the classification, attack priority 3 targets only when a UCAV is unlikely to find a priority 1 or 2 target. This ROE could be compared with standard ROE's regardless of the type of target. One could then incorporate ROE's for cooperative attack, where the number of warheads dispensed depends on the priority of the target.

6. Incorporate cooperative attack. The cooperation discussed in this dissertation was mainly for classification and did not investigate the concept of allowing more than one attack on a target. Further work could be done where a UCAV trades off the chance of attacking another target in its assigned area versus the chance of killing a target found (and perhaps attacked) by another UCAV. This would be an extension of work done by Jacques [28] for the single-warhead case. This development would have some desired level of probability of kill for which multiple warheads may be required.

7. If multiple types of targets are utilized. Incorporate ROE's for cooperative attack. For example: Use two warheads immediately in attacking a priority one target (from the UCAV that found the target if enough warheads are available). For a priority 2 target, use only one warhead from the UCAV that found it and one from another UCAV if it is below a certain probability threshold of finding a priority 1 target. For a priority 3 target, only use a warhead if below a certain probability threshold of finding a priority 1 or 2 target. This is just an example. Maybe instead, we would want to use three different required

probabilities of kill for three types of targets. The possible tie in to the multiple target classification ROE's is obvious.

8. Incorporate moving targets into the analytics. Maybe once a target is found it then has a circular normal distribution for its location. Similarly then, extend the circular normal distribution work by allowing a center other than the origin.

9. Eliminate the assumption that there are more targets than warheads and more false targets than warheads. This assumption is inherent in each scenario with a finite number of targets and/or false targets (with the exception of Scenario 1).

10. Incorporate correlated looks. That is, eliminate the assumption of independence when looking at the same type of target with the same type of sensor. With this assumption we were able to say that the probability of declaring three targets given that there are three targets is $(P_{TR})^3$. Without this assumption, the probability of correctly declaring the first target is P_{TR} but the probability of correctly declaring the second target given we correctly declared the first target is possibly greater than P_{TR}.

Appendix A. Probability of an Exact Number of TA and FTA

We have the following equations for the probability of exactly t target attacks (TA) and f false target attacks (FTA) in normalized time x.

Scenario 1:

$$\begin{aligned} P\left(X_{1,f,x} : t+f < w\right) &= P_{TR} x e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} , && \text{(A.1)} \\ P\left(X_{0,f,x} : t+f < w\right) &= \left(1 - P_{TR} x\right) e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} , && \text{(A.2)} \\ P\left(X_{1,f,x} : t+f = w\right) &= \frac{w P_{TR}}{\lambda_{A_{FT}}} \frac{\gamma\left(w, \lambda_{A_{FT}} x\right)}{\Gamma\left(w\right)} , && \text{(A.3)} \\ P\left(X_{0,f,x} : t+f = w\right) &= \frac{\gamma\left(w, \lambda_{A_{FT}} x\right)}{\Gamma(w)} - P_{TR} \frac{w}{\lambda_{A_{FT}}} \frac{\gamma\left(w+1, \lambda_{A_{FT}} x\right)}{\Gamma\left(w+1\right)} . && \text{(A.4)} \end{aligned}$$

Scenario 2:

$$\begin{aligned} P\left(X_{t,f,x} : t+f < w\right) &= \frac{(\lambda_{A_T} x)^t}{t!} \frac{(\lambda_{A_{FT}} x)^f}{f!} e^{-(\lambda_{A_T} + \lambda_{A_{FT}})x} , && \text{(A.5)} \\ P\left(X_{t,f,x} : t+f = w\right) &= \lambda_{A_T}^t \lambda_{A_{FT}}^{w-t} \binom{w}{t} \frac{\gamma\left(w, \left(\lambda_{A_T} + \lambda_{A_{FT}}\right) x\right)}{\Gamma(w) \left(\lambda_{A_T} + \lambda_{A_{FT}}\right)^w} . && \text{(A.6)} \end{aligned}$$

Scenario 3:

$$P\left(X_{t,f,x} : t+f < w\right) = \binom{N}{t} (P_{TR} x)^t (1 - P_{TR} x)^{N-t} e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} , \quad \text{(A.7)}$$

$$\begin{aligned} \dot{P}\left(X_{t,f,x} : t+f = w\right) = & \binom{N}{t} (P_{TR} x)^t (1 - P_{TR} x)^{N-t} \\ & \times e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} \left(\frac{1}{x} t + \frac{1}{x} f\right) \end{aligned} \quad \text{(A.8)}$$

$$P\left(X_{t,f,x} : t+f < w\right) = \binom{N}{t} \mathbb{A}_u^t (1 - \mathbb{A}_u)^{N-t} e^{-\mathbb{B}_p} \frac{\mathbb{B}_p^f}{f!} , \quad \text{(A.9)}$$

$$\dot{P}\left(X_{t,f,x} : t+f = w\right) = \binom{N}{t} \mathbb{A}_u^t (1 - \mathbb{A}_u)^{N-t} e^{-\mathbb{B}_p} \frac{\mathbb{B}_p^f}{f!} \left(\frac{\mathbb{A}_u'}{\mathbb{A}_u} t + \frac{\mathbb{B}_p'}{\mathbb{B}_p} f\right) . \quad \text{(A.10)}$$

Where

$$\mathbb{A}_u = P_{TR}x \ , \tag{A.11}$$

$$\mathbb{B}_p = \lambda_{A_{FT}}x \ . \tag{A.12}$$

Scenario 4:

$$\begin{aligned} P\left(X_{t,f,x} : t+f < w\right) = & \binom{N}{t}(P_{TR}x)^t(1-P_{TR}x)^{N-t} \\ & \times\binom{M}{f}((1-P_{FTR})x)^f\left(1-(1-P_{FTR})x\right)^{M-f} \ , \end{aligned} \tag{A.13}$$

$$\begin{aligned} \dot{P}\left(X_{t,f,x} : t+f = w\right) = & \binom{N}{t}(P_{TR}x)^t(1-P_{TR}x)^{N-t} \\ & \times\binom{M}{f}((1-P_{FTR})x)^f\left(1-(1-P_{FTR})x\right)^{M-f}\frac{1}{x}(t+f) \ . \end{aligned} \tag{A.14}$$

$$\begin{aligned} P\left(X_{t,f,x} : t+f < w\right) = & \binom{N}{t}\mathbb{A}_u^t(1-\mathbb{A}_u)^{N-t} \\ & \times\binom{M}{f}\mathbb{B}_u^f(1-\mathbb{B}_u)^{M-f} \ , \end{aligned} \tag{A.15}$$

$$\begin{aligned} \dot{P}\left(X_{t,f,x} : t+f = w\right) = & \binom{N}{t}\mathbb{A}_u^t(1-\mathbb{A}_u)^{N-t} \\ & \times\binom{M}{f}\mathbb{B}_u^f(1-\mathbb{B}_u)^{M-f}\left(\frac{\mathbb{A}'_u}{\mathbb{A}_u}t+\frac{\mathbb{B}'_u}{\mathbb{B}_u}f\right) \ . \end{aligned} \tag{A.16}$$

Where

$$\mathbb{A}_u = P_{TR}x \ , \tag{A.17}$$

$$\mathbb{B}_u = (1-P_{FTR})\,x \ . \tag{A.18}$$

Scenario 5:

$$P\left(X_{t,f,\rho} : t+f < w\right) = \binom{N}{t}\mathbb{A}_c^t(1-\mathbb{A}_c)^{N-t}e^{-\mathbb{B}_{cp}}\frac{\mathbb{B}_{cp}^f}{f!} \ , \tag{A.19}$$

$$\dot{P}\left(X_{t,f,\rho} : t+f=w\right) = \binom{N}{t}\mathbb{A}_c^t(1-\mathbb{A}_c)^{N-t}e^{-\mathbb{B}_{cp}}\frac{\mathbb{B}_{cp}^f}{f!}\left(\frac{\mathbb{A}_c'}{\mathbb{A}_c}t+\frac{\mathbb{B}_{cp}}{\mathbb{B}_{cp}'}f\right) . \tag{A.20}$$

Where

$$\mathbb{A}_c = \left(1-e^{-\frac{\rho^2}{2\sigma_T^2}}\right)P_{TR} , \tag{A.21}$$

$$\mathbb{B}_{cp} = \alpha_c\pi\rho^2(1-P_{FTR}) . \tag{A.22}$$

Scenario 6:

$$P\left(X_{t,f,\rho} : t+f<w\right) = \binom{N}{t}\mathbb{A}_c^t(1-\mathbb{A}_c)^{N-t}\binom{M}{f}\mathbb{B}_c^f(1-\mathbb{B}_c)^{M-f} , \tag{A.23}$$

$$\begin{aligned}\dot{P}\left(X_{t,f,\rho} : t+f=w\right) = & \binom{N}{t}\mathbb{A}_c^t(1-\mathbb{A}_c)^{N-t} \\ & \times\binom{M}{f}\mathbb{B}_c^f(1-\mathbb{B}_c)^{M-f}\left(\frac{\mathbb{A}_c'}{\mathbb{A}_c}t+\frac{\mathbb{B}_c'}{\mathbb{B}_c}f\right) .\end{aligned} \tag{A.24}$$

Where

$$\mathbb{A}_c = \left(1-e^{-\frac{\rho^2}{2\sigma_T^2}}\right)P_{TR} , \tag{A.25}$$

$$\mathbb{B}_c = \left(1-e^{-\frac{\rho^2}{2\sigma_{FT}^2}}\right)(1-P_{FTR}) . \tag{A.26}$$

Appendix B. A proof that we can substitute Poisson parameters for T and FT with Poisson parameters for TA and FTA

In [29] a proof is submitted which shows that we can substitute the parameters $(1 - P_{FTR})\alpha A$ for $\alpha_A A$ when determining $P(\mathcal{F}_{0,A})$. This relates the Poisson distribution parameter for FT's to the Poisson distribution parameter for FTA. In this way, we do not have to talk in terms of rate of occurrence of attacks (which is composed of many events). Now we can discuss the rate of occurrence of FT's we encounter. A proof is submitted here which shows we can do the same thing when using multiple warheads. We shall concentrate on FT, but similar things can be done for TA and T.

We have seen previously that for the multiple warhead case, the probability of f FTA's in A (assuming not all warheads are used) is

$$P(\mathcal{F}_{f,A}) = e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} \ ,$$

but if we can say (as was done in the single warhead case) that

$$\lambda_{A_{FT}} = \lambda_{FT}(1 - P_{FTR}) \tag{B.1}$$

then

$$P(\mathcal{F}_{f,A}) = e^{-\lambda_{FT}(1-P_{FTR})x} \frac{(\lambda_{FT}(1 - P_{FTR})x)^f}{(f)!} \ . \tag{B.2}$$

At the same time when we derive $P(\mathcal{F}_{f,A})$ we find

$$\begin{aligned} P(\mathcal{F}_{f,A}) &= P\{FT_{j \geq f,A} \cap NRFT_f\} && \text{(B.3)} \\ &= \sum_{j=f}^{\infty} e^{-\lambda_{FT} x} \frac{(\lambda_{FT} x)^j}{(j)!} \binom{j}{f} (1 - P_{FTR})^f \, (P_{FTR})^{j-f} \ , && \text{(B.4)} \end{aligned}$$

where NR_f indicates that exactly f of the FT's are not recognized (seen as T's). Note that we have assumed here that the multiple events of (mistaking a FT for T) and (mistaking a subsequent FT for T) are independent of each other. The binomial coefficient is the number of combinations possible when only misdiagnosing f FT's out of the j FT's we have actually come across.

So if (B.2) and (B.4) are equal, we can conclude that we can make the substitution described in (B.1) when dealing with a not-all-warheads-used situation.

Theorem 1. *When computing the probability of a certain number of false target attacks given that not all the warheads have been used, we can make the substitution*

$$\lambda_{A_{FT}} = \lambda_{FT}(1 - P_{FTR}) \ .$$

Proof. We will start with an identity and then show that (B.2) and (B.4) are equal

$$\begin{aligned}
e^{\lambda_{FT} P_{FTR} x} &= e^{\lambda_{FT} P_{FTR} x} \\
\Rightarrow \sum_{j=0}^{\infty} \frac{(\lambda_{FT} P_{FTR} x)^j}{(j)!} &= e^{\lambda_{FT} P_{FTR} x} \\
\Rightarrow \sum_{j=f}^{\infty} \frac{(\lambda_{FT} P_{FTR} x)^{j-f}}{(j-f)!} &= e^{\lambda_{FT} P_{FTR} x} \\
\Rightarrow \sum_{j=f}^{\infty} \left[(\lambda_{FT} x)^{j-f} \, \frac{(P_{FTR})^{j-f}}{(j-f)!} \right] &= e^{\lambda_{FT} P_{FTR} x} \\
\Rightarrow \sum_{j=f}^{\infty} \left[e^{-\lambda_{FT} x} \, (\lambda_{FT} x)^j \, \frac{(P_{FTR})^{j-f}}{(j-f)!} \right] &= e^{-\lambda_{FT} x} e^{\lambda_{FT} P_{FTR} x} \, (\lambda_{FT} x)^f \\
\Rightarrow \sum_{j=f}^{\infty} \left[e^{-\lambda_{FT} x} \frac{(\lambda_{FT} x)^j}{(j)!} \frac{j!}{f!(j-f)!} \, (P_{FTR})^{j-f} \right] &= e^{-\lambda_{FT}(1-P_{FTR})x} \frac{(\lambda_{FT} x)^f}{(f)!} \\
\Rightarrow \sum_{j=f}^{\infty} \left[e^{-\lambda_{FT} x} \frac{(\lambda_{FT} x)^j}{(j)!} \binom{j}{f} \, (1 - P_{FTR})^f \, (P_{FTR})^{j-f} \right] &= e^{-\lambda_{FT}(1-P_{FTR})x} \frac{(\lambda_{FT}(1 - P_{FTR})x)^f}{(f)!}
\end{aligned}$$

$\square$

In the case where all the warheads are used, we have seen that

$$P(\mathcal{F}_{f,A}) = \int_0^x e^{-\lambda_{A_{FT}} z} \frac{(\lambda_{A_{FT}} z)^{f-1}}{f-1!} \lambda_{A_{FT}} dz \ ,$$

which becomes (after the substitution in (B.1))

$$P(\mathcal{F}_{f,A}) = \int_0^x e^{-\lambda_{FT}(1-P_{FTR})z} \frac{(\lambda_{FT}(1-P_{FTR})z)^{f-1}}{(f-1)!} \lambda_{FT}(1-P_{FTR})dz \ . \tag{B.5}$$

Whereas a straight derivation leads to

$$P(\mathcal{F}_{f,A}) = \sum_{j=f}^{\infty} \int_0^x e^{-\lambda_{FT}z} \frac{(\lambda_{FT}z)^{j-1}}{(j-1)!} \binom{j-1}{f-1} (1-P_{FTR})^f (P_{FTR})^{(j-1)-(f-1)} \lambda_{FT}(1-P_{FTR})dz \ . \tag{B.6}$$

If (B.5) and (B.6) are equal, then we can make the substitution (B.1) when dealing with an all-warheads-used situation.

Theorem 2. *When computing the probability of a certain number of false target attacks given that all the warheads have been used, we can make the substitution*

$$\lambda_{A_{FT}} = \lambda_{FT}(1-P_{FTR}) \ .$$

Proof. We shall start with an identity and then show that (B.5) and (B.6) are equal

$$\begin{aligned}
\Rightarrow \int_0^x e^{-\lambda_{FT}z} z^{f-1} e^{\lambda_{FT}P_{FTR}z} dz &= \int_0^x e^{-\lambda_{FT}z} e^{\lambda_{FT}P_{FTR}z} z^{f-1} dz \\
\Rightarrow \int_0^x e^{-\lambda_{FT}z} z^{f-1} \sum_{j=0}^{\infty} \frac{(\lambda_{FT}P_{FTR}z)^{(j)}}{(j)!} dz &= \int_0^x e^{-\lambda_{FT}z} e^{\lambda_{FT}P_{FTR}z} z^{f-1} dz \\
\Rightarrow \sum_{j=0}^{\infty} \int_0^x e^{-\lambda_{FT}z} z^{j+f-1} \frac{(\lambda_{FT}P_{FTR})^{(j)}}{(j)!} dz &= \int_0^x e^{-\lambda_{FT}z} e^{\lambda_{FT}P_{FTR}z} z^{f-1} dz \\
\Rightarrow \sum_{j=f}^{\infty} \int_0^x e^{-\lambda_{FT}z} z^{j-1} \frac{(\lambda_{FT}P_{FTR})^{(j-f)}}{(j-f)!} dz &= \int_0^x e^{-\lambda_{FT}(1-P_{FTR})z} z^{f-1} dz \\
\Rightarrow \sum_{j=f}^{\infty} \left[\int_0^x e^{-\lambda_{FT}z} \frac{z^{j-1}}{(j-1)!} \frac{(j-1)!}{(f-1)!(j-f)!} (\lambda_{FT}P_{FTR})^{(j-f)} dz \right] &= \int_0^x e^{-\lambda_{FT}(1-P_{FTR})z} \frac{z^{f-1}}{(f-1)!} dz \\
\Rightarrow \sum_{j=f}^{\infty} \left[\int_0^x e^{-\lambda_{FT}z} \frac{\lambda_{FT}^{j-f} z^{j-1}}{(j-1)!} \binom{j-1}{f-1} (P_{FTR})^{(j-f)} dz \right] &= \int_0^x e^{-\lambda_{FT}(1-P_{FTR})z} \frac{z^{f-1}}{(f-1)!} dz
\end{aligned}$$

$$\Rightarrow \sum_{j=f}^{\infty}\left[\int_0^x e^{-\lambda_{FT}z}\frac{\lambda_{FT}^j z^{j-1}}{(j-1)!}\binom{j-1}{f-1}(1-P_{FTR})^f(P_{FTR})^{(j-f)}dz\right] = \int_0^x e^{-\lambda_{FT}(1-P_{FTR})z}\frac{\lambda_{FT}^f(1-P_{FTR})^f z^{f-1}}{(f-1)!}dz$$

$$\sum_{j=f}^{\infty}\left[\int_0^x e^{-\lambda_{FT}z}\frac{(\lambda_{FT}z)^{j-1}}{(j-1)!}\binom{j-1}{f-1}(1-P_{FTR})^f(P_{FTR})^{(j-1)-(f-1)}\lambda_{FT}(1-P_{FTR})dz\right] = \int_0^x e^{-\lambda_{FT}(1-P_{FTR})z}\frac{(\lambda_{FT}(1-P_{FTR})z)^{f-1}}{(f-1)!}\lambda_{FT}(1-P_{FTR})dz$$

$\square$

Appendix C. Proof of at most one event in infinitesimal area

We wish to determine if the probability of more than one event in Δx as $\Delta x \to \infty$ is negligible. This is a well known result for a Poisson process. We will see if it is also true for the uniform and circular normal distribution. To make this determination, we will follow the same reasoning that Kulkarni [40] uses to show the statement is true for the Poisson distribution.

In that development, they show that the Poisson process $\{N(h) : h \geq 0\}$ has probability masses for $j \geq 2$ events given by

$$P\{N(h) = j\} = o(h), \ j \geq 2 \ , \tag{C.1}$$

where a function $f(h)$ is $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0 \ . \tag{C.2}$$

First we shall examine the Poisson process to demonstrate the methodology. Then we shall apply it to the uniform and circular normal distributions.

C.1 Poisson distribution

First we will find (verify) the probability of one target in Δx and find the limit as $\Delta x \to 0$. We will let this probability equal $f(\Delta x) + o(\Delta x)$,

$$P\{T_{1,\Delta x}\} \quad = \quad f(\Delta x) + o(\Delta x) \ . \tag{C.3}$$

Then

$$\begin{aligned} o(\Delta x) &= P\{T_{1,\Delta x}\} - f(\Delta x) && \text{(C.4)} \\ \Rightarrow & \\ 0 &= \lim_{\Delta x \to 0} \frac{P\{T_{1,\Delta x}\} - f(\Delta x)}{\Delta x} && \text{(C.5)} \\ 0 &= \lim_{\Delta x \to 0} \frac{e^{-\lambda_T \Delta x}\lambda_T \Delta x - f(\Delta x)}{\Delta x} \ . && \text{(C.6)} \end{aligned}$$

For this to be true, $f(0) = 0$ must be true otherwise our limit would be infinity. Therefore, we must also be able to use L'Hopital's rule.

$$
\begin{aligned}
0 &= \lim_{\Delta x \to 0} \frac{e^{-\lambda_T \Delta x} \lambda_T \Delta x - f(\Delta x)}{\Delta x} && \text{(C.7)} \\
0 &= \lim_{\Delta x \to 0} \frac{\frac{d}{d(\Delta x)} \left(e^{-\lambda_T \Delta x} \lambda_T \Delta x - f(\Delta x) \right)}{\frac{d}{d(\Delta x)} \Delta x} && \text{(C.8)} \\
0 &= \lim_{\Delta x \to 0} \frac{\left(-\lambda_T e^{-\lambda_T \Delta x} \lambda_T \Delta x + e^{\lambda_T \Delta x} \lambda_T - f'(\Delta x) \right)}{1} . && \text{(C.9)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f'(\Delta x) &= \lambda_T \ , && \text{(C.10)} \\
\Rightarrow & \\
f(\Delta x) &= \lambda_T \Delta x + c \ . && \text{(C.11)}
\end{aligned}
$$

Because of our initial condition $f(0) = 0$, we see that $c = 0$ and therefore

$$
P\{T_{1,\Delta x}\} = \lambda_T \Delta x + o(\Delta x) \ . \qquad \text{(C.12)}
$$

Similarly, for $P\{T_{j,\Delta x}\}, \ j \geq 2$, we let

$$
P\{T_{j,\Delta x}\} = f(h) + 0(h) \qquad \text{(C.13)}
$$

So then

$$
\begin{aligned}
o(\Delta x) &= P\{T_{j,\Delta x}\} - f(\Delta x) && \text{(C.14)} \\
\Rightarrow & \\
0 &= \lim_{\Delta x \to 0} \frac{P\{T_{j,\Delta x}\} - f(\Delta x)}{\Delta x} && \text{(C.15)} \\
0 &= \lim_{\Delta x \to 0} \frac{e^{-\lambda_T \Delta x} \frac{(\lambda_T \Delta x)^j}{j!} - f(\Delta x)}{\Delta x} . && \text{(C.16)}
\end{aligned}
$$

For this to be true, we must use L'Hopitals rule; hence $f(0) = 0$. Therefore

$$0 = \lim_{\Delta x \to 0} \frac{e^{-\lambda_T \Delta x} \frac{(\lambda_T \Delta x)^j}{j!} - f(\Delta x)}{\Delta x} \tag{C.17}$$

$$0 = \lim_{\Delta x \to 0} \frac{\frac{d}{d(\Delta x)} \left(e^{-\lambda_T \Delta x} \frac{(\lambda_T \Delta x)^j}{j!} - f(\Delta x) \right)}{\frac{d}{d(\Delta x)} \Delta x} \tag{C.18}$$

$$0 = \lim_{\Delta x \to 0} \frac{\left(-\lambda_T e^{-\lambda_T \Delta x} \frac{(\lambda_T \Delta x)^j}{j!} + e^{\lambda_T \Delta x} \frac{j(\lambda_T \Delta x)^{j-1} \lambda_T}{j!} - f'(\Delta x) \right)}{1} . \tag{C.19}$$

Which means

$$0 = \frac{0 + 0 - f'(0)}{1} , \tag{C.20}$$

$$f'(\Delta x) = 0 , \tag{C.21}$$

$$\Rightarrow$$

$$f(\Delta x) = c . \tag{C.22}$$

Because of our initial condition $f(0) = 0$, we see that $c = 0$ and therefore

$$P\{T_{j,\Delta x}\} = o(\Delta x) , \; j \geq 2 . \tag{C.23}$$

C.2 Uniform distribution

Now we will use the same procedure to examine the uniform scenario. We will limit our discussion to a uniform distribution of N targets although the same thing can be done for the uniform distribution of M false targets.

The development for $P\{T_{1,\Delta x}\}$ is the same down to equation (C.5). And the development of $P\{T_{1,\Delta x}\}$, $j \geq 2$, is the same down to equation (C.15). Therefore we will start with these equations.

The probability of one target in Δx as $\Delta x \to \infty$ for the uniform distribution leads to

$$0 = \lim_{\Delta x \to 0} \frac{\binom{N}{1} (\Delta x)^1 (1 - \Delta x)^{N-1} - f(\Delta x)}{\Delta x} . \tag{C.24}$$

Again, we must use L'Hopitals rule for this to be true, therefore $f(0) = 0$ and

$$
\begin{aligned}
0 &= \lim_{\Delta x \to 0} \frac{\frac{d}{d(\Delta x)} \left(\binom{N}{1} (\Delta x)^1 (1-\Delta x)^{N-1} - f(\Delta x) \right)}{\frac{d}{d(\Delta x)} \Delta x} \ , && \text{(C.25)} \\
0 &= \lim_{\Delta x \to 0} \frac{\left(\binom{N}{1} \left((1-\Delta x)^{N-1} - \Delta x (N-1)(1-\Delta x)^{N-2} \right) - f'(\Delta x) \right)}{1} , && \text{(C.26)} \\
0 &= \binom{N}{1} - f'(\Delta x) \ , && \text{(C.27)} \\
\Rightarrow & && \\
f'(\Delta x) &= N \ , && \text{(C.28)} \\
\Rightarrow & && \\
f(\Delta x) &= N \Delta x + c \ . && \text{(C.29)}
\end{aligned}
$$

Since $f(0) = 0$, then $c = 0$, so

$$
P\{T_{1,\Delta x}\} = N \Delta x + o(\Delta x) \tag{C.30}
$$

Similarly, examining $P\{T_{j,\Delta x}\}, \ j \geq 2$ leads us to

$$
0 = \lim_{\Delta x \to 0} \frac{\binom{N}{j} (\Delta x)^j (1-\Delta x)^{N-j} - f(\Delta x)}{\Delta x} \tag{C.31}
$$

Again, we must force the use of L'Hopitals rule for this to be true, therefore $f(0) = 0$ and

$$\begin{aligned}
0 &= \lim_{\Delta x \to 0} \frac{\frac{d}{d(\Delta x)}\left(\binom{N}{j}(\Delta x)^j(1-\Delta x)^{N-j} - f(\Delta x)\right)}{\frac{d}{d(\Delta x)}\Delta x} \ , && \text{(C.32)} \\
&= \lim_{\Delta x \to 0} \left(\binom{N}{j}\left[j(\Delta x)^{j-1}(1-\Delta x)^{N-1} + \right.\right. \\
&\quad \left.\left.(\Delta x)^j(N-j)(1-\Delta x)^{N-j-1}(-1)\right] - f'(\Delta x)\right) \ , && \text{(C.33)} \\
&= 0 - f'(\Delta x) \ , && \text{(C.34)} \\
\Rightarrow & \\
f'(\Delta x) &= 0 \ , && \text{(C.35)} \\
\Rightarrow & \\
f(\Delta x) &= c \ . && \text{(C.36)}
\end{aligned}$$

Since $f(0) = 0$, then $c = 0$, so

$$P\{T_{j,\Delta x}\} = o(\Delta x), \ j \geq 2 \ . \tag{C.37}$$

C.3 Circular Normal

Recall that the probability of the ith target being in a circular area of radius ρ is

$$f_i(\rho, \theta) = \frac{1}{2\pi\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} \ . \tag{C.38}$$

Also recall that the elemental area in the annulus with an inner radius of ρ and a width of dh is $\rho d\theta dh$ so then

$$P\{T_{i^{th},[\rho,\rho+h]}\} = \int_{\rho}^{\rho+h}\int_{0}^{2\pi} \frac{1}{2\pi\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} \rho d\theta dh \ , \tag{C.39}$$

$$= \int_{\rho}^{\rho+h} \frac{1}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} \rho dh \ , \tag{C.40}$$

$$= -\left[e^{-\frac{\rho^2}{2\sigma_T^2}} \right]_{\rho}^{\rho+h} \ , \tag{C.41}$$

$$= e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \ . \tag{C.42}$$

Therefore,

$$P\{T_{1,[\rho,\rho+h]}\} = \binom{N}{1} \left[e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right]^1 \left[1 - \left(e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right) \right]^{N-1} \tag{C.43}$$

Now, to see what the probability is as $h \to 0$, we follow the same method we have used previously in this appendix. That is, we let $P\{\rho, \rho+h\} = f(h) + o(h)$ and compute something very similar to (C.5) (the only difference being we have replaced Δx in (C.5)). We now continue from this point.

$$0 = \lim_{h \to 0} \frac{\left(\binom{N}{1} \left[e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right]^1 \left[1 - \left(e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right) \right]^{N-1} - f(h) \right)}{dh} \tag{C.44}$$

As we have seen so far, we must force the use of L'Hopitals' rule for (C.44) to be true. Therefore, $f(0) = 0$ and

$$0 = \lim_{h \to 0} \frac{\frac{d}{dh} \left(\binom{N}{1} \left[e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right]^1 \left[1 - \left(e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right) \right]^{N-1} - f(h) \right)}{\frac{d}{dh} dh} \tag{C.45}$$

$$
\begin{aligned}
0 &= \lim_{h\to 0}\left(\binom{N}{1}\left(\left[-\frac{-(\rho+h)}{\sigma_T^2}e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}}\right]\left[1-e^{-\frac{\rho^2}{2\sigma_T^2}}+e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}}\right]^{N-1}+\right.\right.\\
&\left[e^{-\frac{\rho^2}{2\sigma_T^2}}-e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}}\right](N-1)\left[1-e^{-\frac{\rho^2}{2\sigma_T^2}}+e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}}\right]^{N-2}\left.\left[\frac{-(\rho+h)}{\sigma_T^2}e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}}\right]\right)\\
&\left.-f'(h)\right)\;, \qquad \text{(C.46)}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(\binom{N}{1}\left(\left[-\frac{-(\rho)}{\sigma_T^2}e^{-\frac{\rho^2}{2\sigma_T^2}}\right]\left[1-e^{-\frac{\rho^2}{2\sigma_T^2}}+e^{-\frac{\rho^2}{2\sigma_T^2}}\right]^{N-1}+\right.\right.\\
&\left[e^{-\frac{\rho^2}{2\sigma_T^2}}-e^{-\frac{\rho^2}{2\sigma_T^2}}\right](N-1)\left[1-e^{-\frac{\rho^2}{2\sigma_T^2}}+e^{-\frac{\rho^2}{2\sigma_T^2}}\right]^{N-2}\left.\left[\frac{-(\rho)}{\sigma_T^2}e^{-\frac{\rho^2}{2\sigma_T^2}}\right]\right)\\
&\left.-f'(0)\right) \qquad \text{(C.47)}
\end{aligned}
$$

$$
0 = N\frac{\rho}{\sigma_T^2}e^{-\frac{\rho^2}{2\sigma_T^2}} - f'(0)\;. \qquad \text{(C.48)}
$$

Therefore,

$$
\begin{aligned}
f'(0) &= N\frac{\rho}{\sigma_T^2}e^{-\frac{\rho^2}{2\sigma_T^2}}\;, \qquad \text{(C.49)}\\
&\Rightarrow\\
f(h) &= N\frac{\rho}{\sigma_T^2}e^{-\frac{\rho^2}{2\sigma_T^2}}h + c\;. \qquad \text{(C.50)}
\end{aligned}
$$

Again, since $f(0) = 0$, then $c = 0$. Recognizing that we were using h in place of $d\rho$ (to try and minimize confusion), we have

$$
P\{T_{1,d\rho}\} = N\frac{\rho}{\sigma_T^2}e^{-\frac{\rho^2}{2\sigma_T^2}}d\rho + o(d\rho)\;. \qquad \text{(C.51)}
$$

Similarly, for $P\{T_{j,h}\},\ j \geq 2$, we start with something similar to (C.15),

$$0 = \lim_{h \to 0} \frac{\left(\binom{N}{j} \left[e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right]^j \left[1 - \left(e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right) \right]^{N-j} - f(h) \right)}{dh} \tag{C.52}$$

Again, we must forcing the use of L'Hopitals' rule. Therefore, $f(0) = 0$ and

$$0 = \lim_{h \to 0} \frac{\frac{d}{dh} \left(\binom{N}{j} \left[e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right]^j \left[1 - \left(e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right) \right]^{N-j} - f(h) \right)}{\frac{d}{dh} dh} \tag{C.53}$$

$$\begin{aligned} 0 \quad = \quad & \lim_{h \to 0} \left(\binom{N}{j} \left(j \left[e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right]^{j-1} \left[-\frac{-(\rho+h)}{\sigma_T^2} e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right] \times \right. \right. \\ & \left[1 - e^{-\frac{\rho^2}{2\sigma_T^2}} + e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right]^{N-j-1} + \left[e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right]^j (N-j) \times \\ & \left. \left. \left[1 - e^{-\frac{\rho^2}{2\sigma_T^2}} + e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right]^{N-j-1} \left[\frac{-(\rho+h)}{\sigma_T^2} e^{-\frac{1}{2}\frac{(\rho+h)^2}{\sigma_T^2}} \right] \right) - f'(h) \right), \end{aligned} \tag{C.54}$$

$$\begin{aligned} 0 \quad = \quad & \left(\binom{N}{j} \left(j \left[e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{\rho^2}{2\sigma_T^2}} \right]^{j-1} \left[-\frac{-(\rho)}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} \right] \left[1 - e^{-\frac{\rho^2}{2\sigma_T^2}} + e^{-\frac{\rho^2}{2\sigma_T^2}} \right]^{N-j-1} + \right. \right. \\ & \left. \left[e^{-\frac{\rho^2}{2\sigma_T^2}} - e^{-\frac{\rho^2}{2\sigma_T^2}} \right]^j (N-j) \left[1 - e^{-\frac{\rho^2}{2\sigma_T^2}} + e^{-\frac{\rho^2}{2\sigma_T^2}} \right]^{N-j-1} \left[\frac{-(\rho)}{\sigma_T^2} e^{-\frac{\rho^2}{2\sigma_T^2}} \right] \right) \\ & \left. -f'(0) \right), \end{aligned} \tag{C.55}$$

$$0 \quad = \quad 0 - f'(0) \tag{C.56}$$

Therefore,

$$
\begin{aligned}
f'(0) &= 0, && \text{(C.57)} \\
\Rightarrow & \\
f(h) &= c\ . && \text{(C.58)}
\end{aligned}
$$

Again, since $f(0) = 0$, then $c = 0$. Recognizing that we were using h in place of $d\rho$, we have

$$P\{T_{j,d\rho}\} = o(d\rho),\ \ j \geq 2\ . \tag{C.59}$$

We note that we often come across expression such as

$$\sum_{i=c}^{N} \binom{N}{i} A^i B^{N-i} \binom{i}{c} \; . \tag{D.1}$$

Recall the binomial theorem states

$$\sum_{i=0}^{N} \binom{N}{i} A^i B^{N-i} \;=\; (A+B)^N \; . \tag{D.2}$$

Our expression would be the normal binomial theorem except the lower limit on the series is not zero and we have multiplied by another binomial coefficient;

$$\begin{aligned}
\sum_{i=c}^{N} \binom{N}{i} A^i B^{N-i} \binom{i}{c} &= \sum_{i=c}^{N} \frac{N!}{i!(N-i)!} \frac{i!}{c!(i-c)!} A^i B^{N-i} \; , && \text{(D.3)}\\
&= \sum_{i=c}^{N} \frac{N!}{(N-i)!} \frac{1}{c!(i-c)!} A^i B^{N-i} \; . && \text{(D.4)}
\end{aligned}$$

Let $k = i - c$, then

$$\begin{aligned}
\sum_{i=c}^{N} \binom{N}{i} A^i B^{N-i} \binom{i}{c} &= \sum_{k=0}^{N-c} \frac{N!}{(N-(k+c))!} \frac{1}{c!k!} A^{k+c} B^{N-(k+c)} \; , && \text{(D.5)}\\
&= \sum_{k=0}^{N-c} \frac{(N-c)!}{k!(N-c-k)!} \frac{N!}{(N-c)!c!} A^{k+c} B^{N-k-c} \; , && \text{(D.6)}\\
&= \sum_{k=0}^{N-c} \binom{N-c}{k} \binom{N}{c} A^k A^c B^{N-c-k} \; , && \text{(D.7)}\\
&= \binom{N}{c} A^c \sum_{k=0}^{N-c} \binom{N-c}{k} A^k B^{N-c-k} \; , && \text{(D.8)}\\
&= \binom{N}{c} A^c \left[A+B\right]^{N-c} \; . && \text{(D.9)}
\end{aligned}$$

Now we also come across

$$\sum_{i=c}^{N} \binom{N}{i} A^i B^{N-i} \binom{i}{c} C^c D^{i-c} = \left(\frac{C}{D}\right)^c \sum_{i=c}^{N} \binom{N}{i} A^i B^{N-i} \binom{i}{c} D^i \ , \tag{D.10}$$

$$= \left(\frac{C}{D}\right)^c \sum_{i=c}^{N} \binom{N}{i} (AD)^i B^{N-i} \binom{i}{c} \ . \tag{D.11}$$

Now using Equation (D.9) we obtain

$$\sum_{i=c}^{N} \binom{N}{i} A^i B^{N-i} \binom{i}{c} C^c D^{i-c} = \left(\frac{C}{D}\right)^c \binom{N}{c} (AD)^c \left[AD + B\right]^{N-c} \ , \tag{D.12}$$

$$= \binom{N}{c} (AC)^c \left[AD + B\right]^{N-c} \ . \tag{D.13}$$

Appendix E. Scenario 3 Markov Example

What follows is an example of the derivation of the equations using the Markov chain method. Along with the Markov model, we will be using variation of parameters to solve the subsequent differential equations.

For Scenario 3, we have

$$\begin{aligned}\dot{P}\left(X_{t,f,x} : t+f < w\right) = \quad & \frac{(N-(t-1))P_{TR}}{(1-P_{TR}x)} P\left(X_{t-1,f,x} : t+f < w\right) + \\ & \lambda_{A_{FT}} P\left(X_{t,f-1,x} : t+f < w\right) - \\ & \left(\frac{(N-t)P_{TR}}{(1-P_{TR}x)} + \lambda_{A_{FT}}\right) P\left(X_{t,f,x} : t+f < w\right) \ ,\end{aligned} \tag{E.1}$$

$$\begin{aligned}P\left(X_{t,f,x} : t+f = w\right) = \quad & \frac{(N-(t-1))P_{TR}}{(1-P_{TR}x)} P\left(X_{t-1,f,x} : t+f < w\right) + \\ & \lambda_{A_{FT}} P\left(X_{t,f-1,x} : t+f < w\right) \ .\end{aligned} \tag{E.2}$$

We will walk thru the sequence and observe the pattern. For brevity sake, we define $P\left(X_{t,f,x}\right) \equiv P\left(X_{t,f,x} : t+f < w\right)$.

State $(t=0, f=0)$ (with initial condition $P\left(X_{0,0,0}\right) = 1$):

$$\dot{P}\left(X_{0,0,x}\right) = -\left(\frac{NP_{TR}}{[1-P_{TR}x]} + \lambda_{A_{FT}}\right) P\left(X_{0,0,x}\right) \ , \tag{E.3}$$

$$\ln P\left(X_{0,0,x}\right) = N \int_0^x \frac{-P_{TR}}{[1-P_{TR}x]} - \int_0^x \lambda_{A_{FT}} + C_1 \ , \tag{E.4}$$

$$\ln P\left(X_{0,0,x}\right) = N \ln(1-P_{TR}x) - \lambda_{A_{FT}} x + C_1 \ , \tag{E.5}$$

$$P\left(X_{0,0,x}\right) = (1-P_{TR}x)^N e^{-\lambda_{A_{FT}} x} C \ , \tag{E.6}$$

$$P\left(X_{0,0,x}\right) = (1-P_{TR}x)^N e^{-\lambda_{A_{FT}} x} \ . \tag{E.7}$$

State $(t=0, f=1)$ (with initial condition $P\left(X_{0,1,0}\right) = 0$):

$$\dot{P}\left(X_{0,1,x}\right) = \lambda_{A_{FT}} P\left(X_{0,0,x}\right) - \left(\frac{NP_{TR}}{[1-P_{TR}x]} + \lambda_{A_{FT}}\right) P\left(X_{0,1,x}\right) \ , \tag{E.8}$$

which has the same homogenous solution as $P(X_{0,0,x})$. Therefore when we use the variation of parameters method, we assume the particular solution takes the form

$$P(X_{0,1,x})_p = v(1-P_{TR}x)^N e^{-\lambda_{A_{FT}}x} C_1 \ , \tag{E.9}$$

where v is some function of x. Then, using variation of parameters, we have

$$\begin{aligned} v'(1-P_{TR}x)^N e^{-\lambda_{A_{FT}}x} &= \lambda_{A_{FT}} P(X_{0,0,x}) = \lambda_{A_{FT}}(1-P_{TR}x)^N e^{-\lambda_{A_{FT}}x} \ , && \text{(E.10)} \\ v' &= \lambda_{A_{FT}} \ , && \text{(E.11)} \\ v &= \lambda_{A_{FT}} x && \text{(E.12)} \end{aligned}$$

Therefore,

$$P(X_{0,1,x}) = (1-P_{TR}x)^N e^{-\lambda_{A_{FT}}x} \lambda_{A_{FT}} x \ . \tag{E.13}$$

Following similar evaluations we can calculate the other solutions. For each state we will give the differential equation and the subsequent solution.

State $(t=0, f=2)$ (with initial condition $P(X_{0,2,0})=0$):

$$\begin{aligned} \dot{P}(X_{0,2,x}) &= \lambda_{A_{FT}} P(X_{0,1,x}) - \left(\frac{NP_{TR}}{[1-P_{TR}x]} + \lambda_{A_{FT}}\right) P(X_{0,2,x}) \ , && \text{(E.14)} \\ P(X_{0,2,x}) &= (1-P_{TR}x)^N e^{-\lambda_{A_{FT}}x} \frac{(\lambda_{A_{FT}}x)^2}{2!} \ . && \text{(E.15)} \end{aligned}$$

State $(t=0, f=3)$ (with initial condition $P(X_{0,3,0})=0$):

$$\begin{aligned} \dot{P}(X_{0,3,x}) &= \lambda_{A_{FT}} P(X_{0,2,x}) - \left(\frac{NP_{TR}}{[1-P_{TR}x]} + \lambda_{A_{FT}}\right) P(X_{0,3,x}) \ , && \text{(E.16)} \\ P(X_{0,3,x}) &= (1-P_{TR}x)^N e^{-\lambda_{A_{FT}}x} \frac{(\lambda_{A_{FT}}x)^3}{3!} \ . && \text{(E.17)} \end{aligned}$$

Now we see the pattern for the increasing f's. For $t=0$, we have

$$P(X_{0,f,x}) = (1-P_{TR}x)^N e^{-\lambda_{A_{FT}}x} \frac{(\lambda_{A_{FT}}x)^f}{f!} \ . \tag{E.18}$$

Now we will look at $t = 1$.

State $(t = 1, f = 0)$ (with initial condition $P(X_{1,0,0}) = 0$):

$$\dot{P}(X_{1,0,x}) = \frac{NP_{TR}}{(1 - P_{TR}x)} P(X_{0,0,x}) - \left(\frac{(N-1)P_{TR}}{(1 - P_{TR}x)} + \lambda_{A_{FT}} \right) P(X_{1,0,x}) \ , \tag{E.19}$$

$$P(X_{1,0,x})_h = (1 - P_{TR}x)^{N-1} e^{-\lambda_{A_{FT}} x} C_2 \ . \tag{E.20}$$

Using variation of parameters,

$$P(X_{1,0,x})_p = v(1 - P_{TR}x)^{N-1} e^{-\lambda_{A_{FT}} x} \ , \tag{E.21}$$

$$v'(1 - P_{TR}x)^{N-1} e^{-\lambda_{A_{FT}} x} = \frac{NP_{TR}}{(1 - P_{TR}x)} (1 - P_{TR}x)^{N} e^{-\lambda_{A_{FT}} x} \ , \tag{E.22}$$

$$v = NP_{TR}x \ . \tag{E.23}$$

Therefore,

$$P(X_{1,0,x}) = NP_{TR}x (1 - P_{TR}x)^{N-1} e^{-\lambda_{A_{FT}} x} \ . \tag{E.24}$$

Using the same method we used for $t = 0$, we determine that

$$P(X_{1,f,x}) = NP_{TR}x (1 - P_{TR}x)^{N-1} e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} \ . \tag{E.25}$$

Similarly, we can determine that

$$P(X_{2,f,x}) = N(N-1) \frac{(P_{TR}x)^2}{2} (1 - P_{TR}x)^{N-2} e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} \ . \tag{E.26}$$

Then, if we continue, we see that

$$P(X_{t,f,x}) = \prod_{i=0}^{t-1} (N-i) \frac{(P_{TR}x)^t}{t!} (1 - P_{TR}x)^{N-t} e^{-\lambda_{A_{FT}} x} \frac{(\lambda_{A_{FT}} x)^f}{f!} \ . \tag{E.27}$$

Appendix F. Attempt to solve time varying differential equation

The Chapman-Kolmogorov equations for our work is of the form

$$\begin{aligned}
\dot{P}\left(X_{t,f,\rho} : t+f < w\right) &= P_{\mathcal{T}_{t-1}}(\rho)P\left(X_{t-1,f,\rho} : t+f < w\right) + \\
&\quad P_{\mathcal{F}_{f-1}}(\rho)P\left(X_{t,f-1,\rho} : t+f < w\right) - \\
&\quad \left(P_{\mathcal{T}_t}(\rho) + P_{\mathcal{F}_f}(\rho)\right) P\left(X_{t,f,\rho} : t+f < w\right) \ , \qquad \text{(F.1)} \\
\dot{P}\left(X_{t,f,\rho} : t+f = w\right) &= P_{\mathcal{T}_{t-1}}(\rho)P\left(X_{t-1,f,\rho} : t+f < w\right) + \\
&\quad P_{\mathcal{F}_{f-1}}(\rho)P\left(X_{t,f-1,\rho} : t+f < w\right) \ . \qquad \text{(F.2)}
\end{aligned}$$

We can put this in a linear algebra form. We will do this by defining the state vector such that

$$\begin{bmatrix} \begin{bmatrix} \dot{P}\left(X_{0,0}\right) \\ \dot{P}\left(X_{0,1}\right) \\ \vdots \\ \dot{P}\left(X_{0,f}\right) \end{bmatrix} \\ \begin{bmatrix} \dot{P}\left(X_{1,0}\right) \\ \dot{P}\left(X_{1,1}\right) \\ \vdots \\ \dot{P}\left(X_{1,f}\right) \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \dot{P}\left(X_{f,0}\right) \\ \dot{P}\left(X_{f,1}\right) \\ \vdots \\ \dot{P}\left(X_{f,f}\right) \end{bmatrix} \end{bmatrix} = [D] \begin{bmatrix} \begin{bmatrix} P\left(X_{0,0}\right) \\ P\left(X_{0,1}\right) \\ \vdots \\ P\left(X_{0,f}\right) \end{bmatrix} \\ \begin{bmatrix} P\left(X_{1,0}\right) \\ P\left(X_{1,1}\right) \\ \vdots \\ P\left(X_{1,f}\right) \end{bmatrix} \\ \vdots \\ \begin{bmatrix} P\left(X_{f,0}\right) \\ P\left(X_{f,1}\right) \\ \vdots \\ P\left(X_{f,f}\right) \end{bmatrix} \end{bmatrix} \qquad \text{(F.3)}$$

where D is of the form

$$D = \begin{bmatrix} [D_0] & 0 & \dots & 0 & 0 \\ [H_0^1] & [D_1] & \dots & 0 & 0 \\ 0 & \begin{bmatrix} H_1 & \ddots \end{bmatrix} & \ddots & 0 & 0 \\ 0 & 0 & \dots & [D_{t-1}] & 0 \\ 0 & 0 & \dots & [H_{t-1}^t] & [D_t] \end{bmatrix} \tag{F.4}$$

and

$$H_{i-2}^{i-1}(s_1) = P_{\mathcal{T}_{i-2}}(s_1)[I]_{(f+1)\times(f+1)} \ , \tag{F.5}$$

$D_i =$

$$\begin{bmatrix} -(P_{\mathcal{T}_i} + P_{\mathcal{F}_0}) & 0 & 0 & \dots & 0 & 0 \\ P_{\mathcal{F}_{1-1}} & -(P_{\mathcal{T}_i} + P_{\mathcal{F}_1}) & 0 & \dots & 0 & 0 \\ 0 & P_{\mathcal{F}_{2-1}} & -(P_{\mathcal{T}_i} + P_{\mathcal{F}_2}) & \dots & 0 & 0 \\ 0 & 0 & P_{\mathcal{F}_{3-1}} & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & -(P_{\mathcal{T}_i} + P_{\mathcal{F}_{f-1}}) & 0 \\ 0 & 0 & 0 & \dots & P_{\mathcal{F}_{f-1}} & -(P_{\mathcal{T}_i} + P_{\mathcal{F}_f}) \end{bmatrix} \tag{F.6}$$

where when $t + f < w$ we have

$$d_{i_{r,r}} = -\left(P_{\mathcal{T}_i} + P_{\mathcal{F}_{r-1}}\right) \ , \tag{F.7}$$

$$d_{i_{r,r-1}} = P_{\mathcal{F}_{r-2}} \ . \tag{F.8}$$

where r goes from 1 to $f+1$, every other element is zero.

Note that in the preceding and subsequent discussion, i indicates the number of TA's (i.e., for the state $X_{1,2}$, $i = 1$). In addition, we use the upper case letters (e.g. D) to represent a matrix. We use the corresponding lower case letter (e.g. d) to represent the element of the matrix denoted by that letter's upper case. A pair of subscripts on the lower case letter indicate a specific element of the matrix (e.g. $d_{3,2}$ would indicate the 3rd row and 2nd column). Since we are dealing with block matrices, we designate the matrix which is a part of the diagonal with a subscript. For example, the matrices which are on the diagonal of the block matrix D are D_i where, again i indicates that that block matrix corresponds to the states that have i TA's. The sub-diagonal block matrices are designated by H_{i-2}^{i-1} with a subscript and superscript. These are just notational conveniences. The subscript indicates which states are being multiplied (in this case, the states with $i-2$ TA's). The superscript indicates the states to whose derivatives this matrix is contributing (in this case, the states with $i-1$ TA's). If the subscript is less than zero, that matrix does not exist.

DeRusso [13] says

$$\dot{x} \quad = \quad A(t)x \tag{F.9}$$

has the solution

$$x(t) \quad = \quad e^{\int_\tau^t A(\lambda)d\lambda} x(\tau) \tag{F.10}$$

if and only if

$$\frac{d}{dt} e^{B(t)} \quad = \quad \frac{dB(t)}{dt} e^{B(t)} \tag{F.11}$$

where

$$B(t) \quad = \quad \int_\tau^t A(\lambda)d\lambda \ . \tag{F.12}$$

This requirement is equivalent to the requirement that

$$A(t_1)A(t_2) \quad = \quad A(t_2)A(t_1) \ . \tag{F.13}$$

Let $T = AB$ and $\tilde{T} = BA$. Further, let $t_{x,y}$ be the element in the xth row and yth column of T. Apply similar definitions for $\tilde{t}$ and $\tilde{T}$; a and A; b and B. Further, let A and B both be $n \times n$ and each is lower triangular with only one sub-diagonal, then

$$t_{i,i} = a_{i,i} b_{i,i} \ , \tag{F.14}$$

$$t_{i+1,i} = a_{i+1,i} b_{i,i} + a_{i+1,i+1} b_{i+1,i} \ , \tag{F.15}$$

$$t_{i+2,i} = a_{i+2,i+1} b_{i+1,i} \ . \tag{F.16}$$

In the same way, we also have

$$\tilde{t}_{i,i} = b_{i,i} a_{i,i} \ , \tag{F.17}$$

$$\tilde{t}_{i+1,i} = b_{i+1,i} a_{i,i} + b_{i+1,i+1} a_{i+1,i} \ , \tag{F.18}$$

$$\tilde{t}_{i+2,i} = b_{i+2,i+1} a_{i+1,i} \ . \tag{F.19}$$

It can be shown that $T = \tilde{T}$ if the following are conditions are met:

1) $a_{i,i} b_{i,i} = b_{i,i} a_{i,i}$ (a necessary condition)

2) The elements of diagonal of A are equal, and the elements of diagonal of B are equal (a sufficient condition). The necessary part is: $a_{i+1,i} b_{i,i} + a_{i+1,i+1} b_{i+1,i} = b_{i+1,i} a_{i,i} + b_{i+1,i+1} a_{i+1,i}$.

3) $a_{i+2,i+1} b_{i+1,i} = b_{i+2,i+1} a_{i+1,i}$

Now in our case (initially), the A and B matrices are the A matrix of the differential equation evaluated at two different times and the elements of the these matrices (a and b) are matrices in and of themselves. Let s_1 and s_2 denote two different times, then we have

$$a_{i,i} = D_{i-1}(s_1) \ , \tag{F.20}$$

$$b_{i,i} = D_{i-1}(s_2) \ , \tag{F.21}$$

$$a_{i,i-1} = H_{i-2}^{i-1}(s_1) \ , \tag{F.22}$$

$$b_{i,i-1} = H_{i-2}^{i-1}(s_2) \ . \tag{F.23}$$

and all other elements are zero.

We must first apply the three requirements to the block matrix found in (F.4), then we apply the three conditions to the resulting matrices.

For example, for the first condition to be met; $a_{i,i}b_{i,i} = b_{i,i}a_{i,i}$ we see from (F.20) thru (F.23) that

$$D_{i-1}(s_1)D_{i-1}(s_2) \quad = \quad D_{i-1}(s_2)D_{i-1}(s_1) \tag{F.24}$$

must be true. To see if this is true, we must apply the three condition to (F.24).

Ultimately, the conditions that must be met boil down to

$$P_{\mathcal{F}_{r-1}}(s_1)P_{\mathcal{F}_r}(s_2) \quad = \quad P_{\mathcal{F}_{r-1}}(s_2)P_{\mathcal{F}_r}(s_1)\ , \tag{F.25}$$

$$P_{\mathcal{T}_i}(s_1)P_{\mathcal{T}_{i-1}}(s_2) \quad = \quad P_{\mathcal{T}_i}(s_2)P_{\mathcal{T}_{i-1}}(s_1) \tag{F.26}$$

These conditions are met for all six scenarios.

Now when $t + f = w$, we have

$$d_{i_{r,r}} \quad = \quad -(P_{\mathcal{T}_i} + P_{\mathcal{F}_{r-1}})\ , \tag{F.27}$$

$$d_{i_{r,r-1}} \quad = \quad P_{\mathcal{F}_{r-2}} \tag{F.28}$$

as r goes from 1 to $f + 1$ except when $i = t$, then

$$d_{t_{f+1,f+1}} \quad = \quad 0\ . \tag{F.29}$$

The final result is that when $t+f = w$ we have two additional conditions which must be met.

$$P_{\mathcal{F}_{f-1}}(s_1)P_{\mathcal{T}_t}(s_2) \quad = \quad P_{\mathcal{F}_{f-1}}(s_2)P_{\mathcal{T}_t}(s_1)\ , \tag{F.30}$$

$$P_{\mathcal{T}_{t-1}}(s_1)P_{\mathcal{F}_f}(s_2) \quad = \quad P_{\mathcal{T}_{t-1}}(s_2)P_{\mathcal{F}_f}(s_1)\ . \tag{F.31}$$

Five of the six scenarios do not meet these conditions. Scenario 2 does.

Therefore, the only scenario for which this method can help us find the probability of a state where $t + f < w$ is Scenario 2. But that is of little help, since the probabilities for this scenario are readily found without this method.

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Vita

Lt Col Doug Decker is a 1981 graduate of Kansas City Christian High School in Shawnee, KS. He subsequently attended college at Cedarville College, Cedarville, OH; Johnson County Community College, Overland Park, KS; and finally finished his undergraduate education at the University of Kansas, Lawrence, KS in Aerospace Engineering in May 1987. At this time Lt Col Decker received his commission in the Air Force through ROTC. He then entered Undergraduate Pilot Training in March 1988. After earning his wings in March 1989, he was assigned as an Instructor Pilot at Reese AFB, TX flying the T-37. After a medical grounding, he was assigned as a Satellite Operations/Engineering Officer at Falcon AFB, CO "flying" GPS satellites. He earned his Masters Degree in astronautical engineering (guidance and control) in 1994 at the Air Force Institute of Technology (AFIT), WPAFB, OH. Since then he has been assigned to the 746th Test Squadron (serving in various capacities testing guidance and navigation systems) and Air Mobility Headquarters, Scott AFB, IL (working in various information warfare capacities). In 2001 he came to AFIT for his PhD program under the direction of Lt Col David R. Jacques.

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14. ABSTRACT

Autonomous wide area search, classification and attack using Unmanned Combat Air Vehicles (UCAVs) is considered. The wide area search and attack scenario is modelled, capturing important problem parameters related to environment, seeker, and munitions. Probabilistic analysis is used to formulate and analytically solve for various probabilities, including the probability of mission success. Two methods are utilized. The first examines the sub-events required for various events to occur. The second utilizes a Markov chain approach. General expressions are first obtained that are applicable to any assumed *a priori* distributions of targets and false targets. These expressions are subsequently applied to a multiple warhead munition/UCAV operating in several multiple target/multiple false target scenarios. Examples of application of the analytically derived results are given for all facets of the system design and operation of Wide Area Search Munitions including the evaluation of cooperation schemes and rules of engagement. The problem is formulated as a control problem, and the possibility of adaptive control based on estimation of environmental parameters is examined.

15. SUBJECT TERMS

Unmanned Air Vehicle, UAV, Unmanned Combat Air Vehicle, UCAV, Search Theory, Wide Area Search Munitions, Cooperative Search, Cooperative Behavior and Control, Cooperative Control

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