9-2006

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AFIT/GAM/ENC/06-05

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THESIS

Presented to the Faculty of the Department of Mathematics and Statisitcs

Graduate School of Engineering and Management

of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the

Requirements for the Degree of

Master of Science

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September 2006

AFIT/GAM/ENC/06-05

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Abstract

We consider the semilinear elliptic equation $\triangle u = p(x)u^{\alpha} + q(x)u^{\beta}$ on a domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, where p and q are nonnegative continuous functions with the property that each of their zeroes is contained in a bounded domain Ω_p or Ω_q, respectively in Ω such that p is positive on the boundary of Ω_p and q is positive on the boundary of Ω_q. For Ω bounded, we show that there exists a nonnegative solution u such that $u(x) \longrightarrow \infty$ as $x \longrightarrow \partial\Omega$ if $0 < \alpha \leq \beta, \beta > 1$, and that such a solution does not exist if $0 < \alpha \leq \beta \leq 1$. For $\Omega = \mathbb{R}^n$, we establish conditions on p and q to guarantee the existence of a nonnegative solution u satisfying $u(x) \longrightarrow \infty$ as $|x| \longrightarrow \infty$ for $0 < \alpha \leq \beta, \beta > 1$, and for $0 < \alpha \leq \beta \leq 1$. For $\Omega = \mathbb{R}^n$ and $0 < \alpha \leq \beta < 1$, we also establish conditions on p and q for the existence and nonexistence of a solution u where u is bounded on $\mathbb{R}^n$.

Acknowledgements

First, I thank God. I said from the beginning that if I made it to graduation, the glory would be completely yours, and here I am.

Next I thank my family. You have been behind me from day one. I may not always show it that well, but I appreciate everything you've done for me, and I love you all very much.

To all of my friends, thank you for always being there for me. Whether it was a few hours at the climbing gym, a late night call from two time zones away, or a late Wednesday night at Applebee's, I had your support and encouragement every time I needed it. To all of those I'm leaving behind in Ohio, I will miss you greatly, and I will do my best to come back and visit.

To the faculty and staff of the United States Air Force Academy, especially those in the Department of Mathematical Sciences, thank you for all of your teaching and mentorship. You made it possible for me to feel overprepared at the start of graduate school. A special thanks to Dr. Lisbeth Schaubroeck for being my undergraduate advisor and for always being there for advice and venting whether the subject was math or something completely different.

My deep thanks go to Dr. Aihua Wood for being my thesis advisor throughout my graduate studies. I have learned so much from you, especially about elliptic theory, in such a short time. Without you, I know that I couldn't have completed the program in a year. I would also like to thank Dr. Matthew Fickus, for taking the time to be on my thesis committee. Lastly, I thank Dr. Alan Lair. Though the seemingly endless revisions in red pen grew to be quite frustrating, I know that you only desired the best for me and the research.

And to anyone I may have left out, please know that I am aware of the impact you've had on my life even if your contribution is not mentioned. Thank you!

David N. Smith

Existence of Large Solutions to Semilinear Elliptic Equations with Multiple Terms

I. *Introduction*

We consider the semilinear elliptic equation

$$\triangle u = p(x)u^{\alpha} + q(x)u^{\beta}, \qquad x \in \Omega \subseteq \mathbb{R}^n, \ n \geq 3, \tag{1}$$

where for $x = (x_1, x_2, \ldots, x_n)$, $\triangle u = \partial_{x_1 x_1} + \partial_{x_2 x_2} + \ldots + \partial_{x_n x_n}$, and Ω is an open, connected set in $\mathbb{R}^n$. Equations such as these are found in the study of steady state diffusion type problems, the study of the subsonic motion of gases [17], the electric potential in some bodies [15], and Riemannian geometry [6]. In addition, we require the functions p and q to be nonnegative and continuous on $\overline{\Omega}$. We consider two cases, the superlinear/mixed ($0 < \alpha \leq \beta, \beta > 1$) case and the sublinear ($0 < \alpha \leq \beta \leq 1$) case. Also, we provide conditions on p and q which guarantee that (1) has a nonnegative solution u such that $u(x) \longrightarrow \infty$ as $x \longrightarrow \partial\Omega$. Such functions are called large solutions of (1) on Ω. If $\Omega = \mathbb{R}^n$, then such functions are called entire large solutions. For the sublinear case, we also consider the existence and nonexistence of a solution u such that u is bounded.

Very little work has been done in reference to the multi-term equation we consider, but much study has been conducted for the single-term equation

$$\triangle u = p(x)u^{\gamma}. \tag{2}$$

The multi-term equation is similar to the single-term equation, but it also presents some challenging differences. Before we discuss the results established in this work, let us take a look at some of the previous work that has been done in this field.

1.1 Background

Problems similar to the one we consider in this work have been under study for many years. The following is an attempt to summarize some of the accomplishments made by others in this area, which has led to our study.

In 1916, Bieberbach [4] first studied large solutions of the semilinear elliptic boundary valued problem

$$\begin{aligned} \triangle u &= f(u), \;\; x \in \Omega, \\ u(x) &\longrightarrow \infty \;\; \text{as} \;\; x \longrightarrow \partial\Omega \end{aligned} \tag{3}$$

for the case where $f(u) = e^u$. It was shown in [4] that (3) has a unique classical large solution in this case in a bounded domain with smooth boundary in $\mathbb{R}^2$. In 1943, Rademacher [19] extended the result to smooth bounded domains in $\mathbb{R}^3$. Necessary and sufficient conditions on f for the existence of solutions to (1) for bounded domains in $\mathbb{R}^n$ were established by Keller [8] and Osserman [16] in 1957. They proved that (3) has a large solution on Ω if and only if the function f satisfies

$$\int_1^\infty \left[\int_0^s f(t)dt\right]^{-1/2} ds < \infty. \tag{4}$$

Later, the asymptotic behavior of the solutions in bounded domains in $\mathbb{R}^n$ was studied by Lazer and McKenna [14].

Bandle and Marcus [3] showed that $\triangle u = g(x, u)$ has a unique large positive solution for bounded and unbounded domains. Notice that this is a more general equation that includes (2), where $g(x, u) = p(x)u^\gamma$, $\gamma > 1$, and $p(x)$ is a positive continuous function in $\overline{\Omega}$ such that p and $\frac{1}{p}$ are bounded. They also proved that the equation

$$\triangle u = p(x)f(u) \tag{5}$$

has a positive large solution provided that the function f satisfies (4) and the function p is continuous and strictly postive on $\overline{\Omega}$. In addition, they studied the asymptotic behavior of such solutions.

Lair [10] showed that the same results hold for (5) if the function p is allowed to vanish on large parts of Ω including its boundary. He also required that the function f be nondecreasing on $[0, \infty)$. Proano [18] extended these results by requiring a weaker condition on the function f. He gave conditions for the existence of large solutions to (5) provided that the function f is nonnegative on $[0, \infty)$ and satisfies the inequality

$$g_1 \leq f \leq g_2, \tag{6}$$

where the functions g_1 and g_2 are continuous and nondecreasing on $[0, \infty)$ with $g_1(0) = 0$, $g_2(0) = 0$, and $g_1(s)$, $g_2(s) > 0$ for $s > 0$, and where p is nonnegative and continuous on $\overline{\Omega}$.

Cheng and Ni [6] provided results for the superlinear ($\gamma > 1$) case of (2), where p is nonnegative and smooth. They proved that (2) has a large solution on a bounded domain Ω if p is strictly positive on $\partial\Omega$. Then, requiring that there exists $m > 2$ such that $|x|^m p(x)$ is bounded for large $|x|$ and that the function p meets a positiveness condition, they proved that (2) has a unique postive entire solution. Asymptotic behavior of the solution near ∞ was also characterized in their results.

Lair and Wood [12] proved the existence of large solutions for a bounded domain Ω under a more relaxed condition for p when compared to the conditions of Cheng and Ni [6] and Bandle and Marcus [3]. More specifically, for the bounded domain, they allowed p to be zero on large portions of $\overline{\Omega}$, including $\partial\Omega$, a weaker requirement than those of [3] or [6] where p is either taken to be either positive and continuous on $\overline{\Omega}$ or p is required to be positve on $\partial\Omega$. Lair and Wood [12] also relaxed the conditions on p for the existence of an entire large solution when compared to the conditions of Cheng and Ni [6]. Lair and Wood required that

$$\int_0^\infty r\phi(r)dr < \infty, \tag{7}$$

where $\phi(r) = \max\limits_{|x|=r} \; p(x)$.

For the sublinear case of (2), very few results are known. Brezis and Kamin [5] gave necessary and sufficient conditions on p for the existence of a bounded solution when $p(x) \leq 0$. Kusano and Oharu [9] studied equations of the form $\triangle u = f(x, u)$ where f is allowed to take both positive and negative values. They provided sufficient conditions for the existence of an entire solution that decays to zero at infinity. In [13], Lair and Wood provided existence and nonexistence results for large solutions for the sublinear ($0 < \gamma \leq 1$) case of (2). For the radial case, where $\triangle u = p(|x|)u^\gamma$, they prove that an entire large solution for (2) exists if and only if

$$\int_0^\infty rp(r)dr = \infty. \tag{8}$$

In addition, Lair and Wood [13] proved that for a bounded domain Ω, (2) has no positive large solution in the sublinear case when p is continuous on $\overline{\Omega}$. They also established existence and nonexistence results for enitre bounded solutions for the sublinear case. For the existence of a nonnegative entire bounded solution in $\mathbb{R}^n$ to (2) they require that (8) hold and that the function p be locally Hölder continuous. As a nonexistence result, Lair and Wood [13] prove that (2) has no nonnegative entire bounded solution in $\mathbb{R}^n$ if

$$\int_0^\infty r \min_{|x|=r} p(x)dr = \infty. \tag{9}$$

In this work, we explore how many results for the single-term equation can be extended to the multi-term equation and establish similar conditions for the existence and nonexistence of large solutions to (1). The only other results known to us are by Lair and Wood [11]. They considered the radial case of (1) for $1 < \alpha \leq \beta$. Our results include their results as a special case. For a bounded domain Ω the results of [12] extend to the multi-term equation for both the superlinear ($1 < \alpha \leq \beta$) and mixed ($0 < \alpha \leq 1 < \beta$) cases. For the existence of a large solution to (1) on a bounded domain Ω in the superlinear and mixed cases we require the same conditions as in [12] and require that the function q meet the same conditions required of p. Similarly, for the superlinear and mixed cases the conditions for entire large solutions to (2) presented in [12] also extend to the multi-term equation. We require that (8) hold and

$$\int_0^\infty r\psi(r)dr < \infty, \tag{10}$$

where $\psi(r) = \max\limits_{|x|=r} \ q(x)$.

We also extend many of the results of [13] to the sublinear ($0 < \alpha \leq \beta \leq 1$) case of (1). For the radial case of (1), where $\triangle u = p(|x|)u^\alpha + q(|x|)u^\beta$ we prove that an entire large solution for the sublinear case exists if and only if (9) holds and

$$\int_0^\infty rq(r)dr = \infty. \tag{11}$$

We also prove that (1) has no positive large solution for a bounded domain in the sublinear case when p and q are continuous on $\overline{\Omega}$. In addition, we extend the results of [13] for entire bounded solutions in the sublinear case. We prove that (1) has a nonnegative entire bounded solution in $\mathbb{R}^n$ in the sublinear case if (8) and (11) hold and if the functions p and q are locally Hölder continuous. Also, we prove that (1) has no nonnegative entire bounded solution in $\mathbb{R}^n$ if (10) holds or if

$$\int_0^\infty r \min_{|x|=r} q(x)dr = \infty. \tag{12}$$

We now examine the underlying elliptic theory that is used to prove our main results.

1.2 Preliminaries

The first concept we present is the idea of barrier methods, also known as upper/lower solution methods. We present both the definitions of an upper and a lower solution and the corresponding theorem, which we will use to prove several of our main results.

Definition 1 *An upper solution to the following boundary value problem*

$$\triangle u = p(x)f(u), \quad x \in \Omega \tag{13}$$
$$u(x) = g(x), \quad x \in \partial\Omega,$$

is a function $\overline{u}$ satisfying

$$\triangle \overline{u} \le p(x) f(\overline{u}), \quad x \in \Omega$$
$$\overline{u}(x) \ge g(x), \quad x \in \partial\Omega.$$

A lower solution to (13) is a function $\underline{u}$ satisfying

$$\triangle \underline{u} \ge p(x) f(\underline{u}), \quad x \in \Omega$$
$$\underline{u}(x) \le g(x), \quad x \in \partial\Omega,$$

Theorem 2 *(Theorem 2.3.1 of [21]) Let ϕ be an upper solution and ξ a lower solution with $\xi \le \phi$ on Ω to Eq. (13). Then, there exists a solution u to (13) with $\xi \le u \le \phi$.*

The upper/lower solution method can also be extended for use in proving the existence of entire bounded solutions. Next, we state a useful variation of the standard maximum principle argument from elliptic theory.

Theorem 3 *(Theorem 3.3 of [7]) Let L be a linear elliptic differential operator of the form*

$$Lu = a_{ij}(x) D_{ij} u + b_i D_i u + c(x) u, \qquad a_{ij} = a_{ji},$$

where $x = (x_1, x_2, \dots, x_n)$ in $\Omega \subseteq \mathbb{R}^n$ with $c(x) \le 0$ in Ω. Suppose that u and v are functions in $C^2(\Omega) \cap C(\overline{\Omega})$ satisfying $Lu \ge Lv$ in Ω and $u \le v$ on $\partial\Omega$. Then, $u \le v$ in Ω.

Now, the Laplacian is a linear elliptic differential operator. Thus, we can let $L = \triangle$ in the above theorem, which will be useful in proving our main results later. We now present the Arzela-Ascoli Theorem, which we will use in proving one of our main results.

Definition 4 *A subset K of a normed space X is called compact if every sequence of points in K has a convergent subsequence in X to an element of K. Furthermore, a subset K of X is called precompact in X if its closure, $\overline{K}$, in the norm topology of X is compact.*

Theorem 5 *(Theorem 1.34 of [1]) (Arzela-Ascoli Theorem) Let Ω be a bounded domain in $\mathbb{R}^n$. A subset K of $C(\overline{\Omega})$ is precompact in $C(\overline{\Omega})$ if:*

(i) There exists $M \geq 0$ such that $|\phi(x)| \leq M$ for every $\phi \in K$ and $x \in \Omega$ (i.e. K is bounded), and

(ii) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|\phi(x) - \phi(y)| < \varepsilon$ for all $\phi \in K$, $x, y \in \Omega$, and $|x - y| < \delta$ (i.e. K is equicontinuous).

We now present two very useful concepts in elliptic theory. First, we state what it means for a bounded domain Ω to have C^2-boundary.

Definition 6 *A bounded domain $\Omega \subseteq \mathbb{R}^n$ has C^2-boundary if at each point $x_0 \in \partial\Omega$ there exists a ball $B = B(x_0, R)$ centered at x_0 with radius R and a one-to-one mapping ω of B onto $\Omega' \subseteq \mathbb{R}^n$ such that:*

(i) $\omega(B \cap \Omega) \subseteq \mathbb{R}^n_+$;

(ii) $\omega(B \cap \partial\Omega) \subseteq \partial\mathbb{R}^n_+$;

(iii) $\omega \in C^2(B)$, $\omega^{-1} \in C^2(\Omega')$.

We will also need the concepts of Hölder continuity and the Hölder space $C^{2+\lambda}(\Omega)$. This space is important to us because we will later show that our solutions are in $C^{2+\lambda}$.

Definition 7 *Let x_0 be a point in $\mathbb{R}^n$ and f a function defined on a bounded open set Ω containing x_0. For $0 < \lambda < 1$, we say that f is Hölder continuous with exponent λ at x_0 if*

$$[f]_{\lambda;x_0} \equiv \sup_{x\in\Omega} \frac{|f(x) - f(x_0)|}{|x - x_0|^\lambda} < \infty,$$

in which case we call it the λ-Hölder coefficient of f at x_0 with respect to Ω. Furthermore, we say that f is uniformly Hölder continuous with exponent λ in Ω if

$$[f]_{\lambda;\Omega} \equiv \sup_{x,y\in\Omega,\ x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\lambda} < \infty, \quad 0 < \alpha \leq 1.$$

Definition 8 *The Hölder space $C^{2+\lambda}(\Omega)$ is a subspace of $C^2(\Omega)$ consisting of functions whose second order partial derivatives are uniformly Hölder continuous with exponent λ.*

The next results we present are from the important theory of Sobolev spaces which will be vital in later demonstrating that the standard bootstrap argument can be used to prove that our solutions are classical solutions of Eq. (1) on Ω. As we will show later, the bootstrap argument makes use of Sobolev imbeddings in order to show that our solutions are truly $C^{2+\lambda}$. In other words, the bootstrap argument shows that our solutions are sufficiently smooth on Ω. In order to present the necessary theory, we must first define the concept of an imbedding.

Definition 9 *(Definition 1.25 of [1]) We say the normed space X is imbedded in the normed space Y, and we write $X \longrightarrow Y$ to designate this imbedding, provided that*

(i) X is a vector subspace of Y, and

(ii) the identity operator I defined on X into Y by $Ix = x$ for all $x \in X$ is continuous.

We are now almost ready to present the Sobolev Imbedding Theorem. However, we must first define the spaces that are involved in the statement of the theorem.

Definition 10 *Let u be locally integrable in Ω and let $\eta = (\eta_1, \ldots \eta_n)$, $|\eta| = \eta_1 + \ldots + \eta_n$. Then, a locally integrable function v is called the ηth weak derivative of u if it satisfies*

$$\int_\Omega \varphi v dx = (-1)^{|\eta|} \int_\Omega u D^\eta \varphi dx \quad \text{for all } \varphi \in C_0^{|\eta|}(\Omega),$$

where $C_0^{|\eta|}(\Omega)$ is the space of functions in $C^{|\eta|}(\Omega)$ with compact support. Furthermore, we call a function k-times weakly differentiable if all its weak derivatives exist for orders up to and including k.

Definition 11 *The Sobolev space $W^{m,p}(\Omega)$ is the Banach space defined by*

$$W^{m,p}(\Omega) = \{u \in L^p : D^\eta u \in L^p(\Omega) \text{ for all } |\eta| \ \leq m\}$$

where $\eta = (\eta_1, \ldots \eta_n)$, $|\eta| = \eta_1 + \ldots + \eta_n$, and the derivatives $D^\eta u$ are weakly differentiable.

The norm in the Sobolev space $W^{m,p}(\Omega)$ is given by

$$\|u\|_{W^{m,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \leq m} |D^\alpha u|^p dx \right)^{1/p}. \tag{14}$$

Now, we present the first chain of imbeddings that we will need to use later in the standard bootstrap argument. From [1] we have the chain of imbeddings

$$W_0^{m,p}(\Omega) \longrightarrow W^{m,p}(\Omega) \longrightarrow L^p(\Omega),\ m \in \mathbb{Z},\ m \geq 1,\ 1 \leq p < \infty$$

where $W_0^{m,p}(\Omega)$ is a Sobolev space of functions in $W^{m,p}(\Omega)$ with compact support, and $L^p(\Omega)$ is the classical Banach space of measurable functions on Ω that are p-integrable, $p \geq 1$.

We must present three more definitions before we can state the Sobolev Imbedding Theorem. We define what it means for a domain to satisfy the cone condition and to satisfy the strong local Lipschitz condition. We also define the space $C_B^j(\Omega)$, a subspace of $C^j(\Omega)$.

Definition 12 *(Definition 4.6 of [1]) The domain Ω satisfies the cone condition if there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C.*

Definition 13 *(Definition 4.9 of [1]) The domain Ω satisfies the strong local Lipschitz condition if there exist positive numbers δ and M, a locally finite open cover $\{U_j\}$ of $\partial\Omega$, and, for each j, a real-valued function f_j of $n-1$ variables such that:*

(i) For some finite R, every collection of $R+1$ of the sets U_j has an empty intersection.

(ii) For every pair of points $x, y \in \Omega_\delta$ such that $|x-y| \leq \delta$, there exists j such that

$$x, y \in V_j \equiv \{x \in U_j : dist(x, \partial U_j) > \delta\}.$$

(iii) Each function f_j satisfies a Lipschitz condition with constant M; that is, if $\beta = (\beta_1, \ldots, \beta_{n-1})$ and $\rho = (\rho_1, \ldots, \rho_{n-1})$ in $\mathbb{R}^{n-1}$, then

$$|f(\beta) - f(\rho)| \leq M|\beta - \rho|.$$

(iv) For some Cartesian coordinate system $(\zeta_{j,1},\ldots,\zeta_{j,n})$ in U_j, $\Omega\cap U_j$ is represented by the inequality

$$\zeta_{j,n} < f_j(\zeta_{j,1},\ldots,\zeta_{j,n}).$$

If Ω is bounded, the above conditions reduce to the condition that Ω should have a locally Lipschitz boundary, that is, each point x on the boundary of Ω should have a neighborhood U_x whose intersection with $\partial\Omega$ should be the graph of a Lipschitz continuous function.

Definition 14 *The space of bounded continuous functions $C_B^j(\Omega)$ is the set of all functions $u\in C^j(\Omega)$ for which $D^\alpha u$ is bounded on Ω for $|\alpha|\leq j$. Furthermore, $C_B^j(\Omega)$ is a Banach space with norm given by*

$$\|u\|_{C_B^j(\Omega)} = \max_{|\alpha|\leq j}\sup_{x\in\Omega}|D^\alpha u(x)|.$$

With the preceding definitions in mind, we now present the Sobolev Imbedding Theorem.

Theorem 15 *(Theorem 4.12 of [1]) Let Ω be a domain in $\mathbb{R}^n$, and for $1\leq k\leq n$, let Ω_k be the intersection of Ω with a plane of dimension k in $\mathbb{R}^n$. (If $k=n$, then $\Omega_k=\Omega$.) Let $j\geq 0$ and $m\geq 1$ be integers and let $1\leq p<\infty$.*

PART I *Suppose Ω satisfies the cone condition.*

Case A *If either $mp>n$ or $m=n$ and $p=1$, then*

$$W^{j+m,p}(\Omega)\longrightarrow C_B^j(\Omega).$$

Moreover, if $1\leq k\leq n$, then

$$W^{j+m,p}(\Omega)\longrightarrow W^{j,q}(\Omega_k), \quad \text{for } p\leq q<\infty,$$

and, in particular,

$$W^{m,p}(\Omega)\longrightarrow L^q(\Omega), \quad \text{for } p\leq q<\infty.$$

Case B *If $1 \leq k \leq n$ and $mp = n$, then*

$$W^{j+m,p}(\Omega) \longrightarrow W^{j,q}(\Omega_k), \quad \text{for } p \leq q < \infty,$$

and, in particular,

$$W^{m,p}(\Omega) \longrightarrow L^q(\Omega), \quad \text{for } p \leq q < \infty.$$

Case C *If $mp < n$ and either $n - mp < k \leq n$ or $p = 1$ and $n - m \leq k \leq n$, then*

$$W^{j+m,p}(\Omega) \longrightarrow W^{j,q}(\Omega_k), \quad \text{for } p \leq q \leq p^* = \frac{kp}{n - mp}.$$

In particular,

$$W^{m,p}(\Omega) \longrightarrow L^q(\Omega), \quad \text{for } p \leq q \leq p^* = \frac{np}{n - mp}.$$

The imbedding constants for the imbeddings above depend only on n, m, p, q, j, k, and the dimensions of the cone C in the cone condition.

PART II *Suppose Ω satisfies the strong local Lipschitz condition. Then, the target space $C_B^j(\Omega)$ of the first imbedding above can be replaced with the smaller space $C^j(\overline{\Omega})$ and the imbedding can be further refined as follows:*

If $mp > n > (m-1)p$, then

$$W^{j+m,p}(\Omega) \longrightarrow C^{j,\lambda}(\overline{\Omega}), \quad \text{for } 0 < \lambda \leq m - \frac{n}{p},$$

and if $n = (m-1)p$, then

$$W^{j+m,p}(\Omega) \longrightarrow C^{j,\lambda}(\overline{\Omega}), \quad \text{for } 0 < \lambda < 1.$$

Also, if $n = m - 1$ and $p = 1$, then the above imbedding holds for $\lambda = 1$ as well.

PART III *All of the imbeddings in Parts A and B are valid for arbitrary domains Ω if the W-space undergoing the imbedding is replaced with the corresponding W_0-space.*

With all of the preceding results in hand, we now have the necessary tools to prove our results.

II. Main Results

In this section we state and prove our results. Through the course of this work, we often require the functions p and q to satisfy the following *circumferentially positive (c-positive)* condition:

Definition 16 *A function p is c-positive on a domain Ω if for any $x_0 \in \Omega$ satisfying $p(x_0) = 0$, there exists a domain Ω_0 such that $x_0 \in \Omega_0, \overline{\Omega}_0 \subset \Omega$, and $p(x) > 0$ for all $x \in \partial\Omega_0$. The function p is c-positive on $\mathbb{R}^n$ if for any $x_0 \in \mathbb{R}^n$ such that $p(x_0) = 0$, there exists a domain Ω_0 such that $x_0 \in \Omega_0$ and $p(x) > 0$ for all $x \in \partial\Omega_0$.*

We now state our first result, which extends Theorem 1 of [12] to the multi-term equation.

2.1 Superlinear/Mixed Case ($0 < \alpha \leq \beta, \beta > 1$)

Theorem 17 *Suppose Ω is a bounded domain in $\mathbb{R}^n, n \geq 3$, with C^2 boundary, and $p, q \in C(\overline{\Omega})$ are nonnegative and c-positive. Then, (1) has a large positive solution in Ω if $\beta > 1$ and $0 < \alpha \leq \beta$.*

Proof. We know from Lair and Wood [12] that for $k \in \mathbb{N}$ there exists a unique positive classical solution to the boundary value problem

$$\begin{aligned} \triangle v_k &= q(x) v_k^{\beta}, \quad x \in \Omega, \\ v_k(x) &= k, \quad x \in \partial\Omega. \end{aligned} \tag{15}$$

Clearly, $\triangle v_k = q(x) v_k^{\beta} \leq p(x) v_k^{\alpha} + q(x) v_k^{\beta}$. Now, we also know from Proposition 1 of Lair [10] that for $k \in \mathbb{N}$ there exists a unique nonnegative classical solution to the boundary value problem

$$\begin{aligned} \triangle w_k &= (p(x) + q(x))(w_k^{\alpha} + w_k^{\beta}), \quad x \in \Omega, \\ w_k(x) &= k, \quad x \in \partial\Omega. \end{aligned} \tag{16}$$

Then,

$$\begin{aligned}
\triangle w_k &= (p(x)+q(x))(w_k^\alpha + w_k^\beta) \\
&= p(x)w_k^\alpha + p(x)w_k^\beta + q(x)w_k^\alpha + q(x)w_k^\beta \\
&\geq p(x)w_k^\alpha + q(x)w_k^\beta.
\end{aligned}$$

Thus by Theorem 3, we know that $w_k \leq v_k$ for all $k \in \mathbb{N}$. By letting $\overline{u}_1 = v_1$ and $\underline{u}_1 = w_1$, we know by Theorem 2 that there exists a nonnegative classical solution u_1 of the boundary value problem

$$\begin{aligned}
\triangle u_1 = p(x)u_1^\alpha + q(x)u_1^\beta, \quad x \in \Omega, \\
u_1(x) = 1, \quad x \in \partial\Omega,
\end{aligned} \tag{17}$$

such that $w_1 = \underline{u}_1 \leq u_1 \leq \overline{u}_1 = v_1$. Similarly, if we let $\overline{u}_2 = v_2$ and $\underline{u}_2 = u_1$, we know there exists a nonnegative classical solution u_2 of the boundary value problem

$$\begin{aligned}
\triangle u_2 = p(x)u_2^\alpha + q(x)u_2^\beta, \quad x \in \Omega, \\
u_2(x) = 2, \quad x \in \partial\Omega,
\end{aligned} \tag{18}$$

such that $w_1 \leq u_1 = \underline{u}_2 \leq u_2 \leq \overline{u}_2 = v_2$. Continuing this reasoning, we have that, for $k \geq 2$, there exists a nonnegative classical solution u_k of the boundary value problem

$$\begin{aligned}
\triangle u_k = p(x)u_k^\alpha + q(x)u_k^\beta, \quad x \in \Omega, \\
u_k(x) = k, \quad x \in \partial\Omega.
\end{aligned} \tag{19}$$

such that $w_1 \leq u_{k-1} \leq u_k \leq v_k$. By construction, the sequence $\{u_k\}$ is monotone. Furthermore, we know from [12] that the sequence $\{v_k\}$ converges on Ω to a large solution of $\triangle v = q(x)v^\beta$ on Ω. It follows that $w_1 \leq u_{k-1} \leq u_k \leq v$. Thus, u_k is bounded. Therefore, since the sequence $\{u_k\}$ is monotone and bounded, it converges on Ω to some function u. We now apply the standard bootstrap argument from [22] to prove that the function $u(x)$ is indeed a solution to (1).

Let $x_0 \in \Omega \subseteq \mathbb{R}^n$, and let $B(x_0, r)$ be a ball centered at x_0 whose radius r is chosen such that $B(x_0, r) \subseteq \Omega$. Let ψ be a C^∞ function which is equal to 1 on $\overline{B(x_0, \frac{r}{2})}$ and zero off $B(x_0, r)$. We have

$$\triangle(\psi u_k) = 2\nabla\psi \cdot \nabla u_k + q_k, \quad k \geq 1, \tag{20}$$

where

$$q_k = u_k \triangle\psi + \psi \triangle u_k \tag{21}$$

is a term whose L^∞ norm is bounded independently of k on $B(x_0, r)$. We therefore have

$$\psi u_k \triangle(\psi u_k) = A_k \cdot \nabla(\psi u_k) + s_k \tag{22}$$

where $A_k = 2u_k \nabla\psi$ and $s_k = \psi u_k q_k - u_k \left[2u_k \nabla\psi \cdot \nabla\psi\right]$ are bounded independently of k. Now, integrating (22) over $B(x_0, r)$ we have

$$\begin{aligned}
\int_{B(x_0,r)} |\nabla(\psi u_k)|^2 dx &= -\int_{B(x_0,r)} \left[A_k \cdot \nabla(\psi u_k) + s_k\right] dx \\
&\leq \overline{c}_1 \left(\int_{B(x_0,r)} |A_k||\nabla(\psi u_k)| dx \right) + c_2 \\
&\leq c_1 \left(\int_{B(x_0,r)} |\nabla(\psi u_k)|^2 dx \right)^{\frac{1}{2}} + c_{2,}
\end{aligned}$$

where $\overline{c}_1$, c_1, and c_2 are some constants independent of k. Hence, we have that

$$\|\nabla(\psi u_k)\|^2_{L^2(B(x_0,r))} \leq c_1^2 + 2c_2. \tag{23}$$

From this, it follows that the $L^2(B(x_0, r))$-norm of $|\nabla(\psi u_k)|$ is bounded independently of k. Hence, the $L^2(B(x_0, \frac{r}{2}))$-norm of $|\nabla u_k|$ is bounded independently of k. Similarly, letting ψ_1 be a C^∞ function which is equal to 1 on $\overline{B(x_0, \frac{r}{4})}$ and zero off $B(x_0, \frac{r}{2})$, we may show that the $W^{2,2}(B(x_0, \frac{r}{4}))$-norm of $|\nabla u_k|$ is bounded independently of k. It then follows from the Sobolev Imbedding Theorem that the $L^q(B(x_0, \frac{r}{4}))$-norm of $|\nabla u_k|$ is bounded independently of k for $q = \frac{2n}{n-2}$.

Continuing this line of reasoning we arrive at a number $r_1 > 0$ such that there is a subsequence of $\{u_k\}_1^\infty$, which we may assume is still the sequence itself, which converges in $C^{1+\lambda}(\overline{B(x_0, r_1)})$, for some positive number $\alpha < 1$.
Let ψ be a C^∞ function which is equal to 1 on $\overline{B(x_0, \frac{r_1}{2})}$ and zero off $B(x_0, r_1)$. Then

$$\triangle(\psi u_k) = 2\nabla\psi \cdot \nabla u_k + \widehat{q}_k, \tag{24}$$

where $\widehat{q}_k$ is given in (24). Now, we consider two cases regarding the regularity of the functions $p(x)$ and $q(x)$.

Case 1: $p(x), q(x) \in C^\infty(\Omega)$. The right-hand side of (27) converges in $C^\lambda(\overline{B(x_0, r_1)})$. Hence, by Schauder theory (See [21]), $\{\psi u_k\}_1^\infty$ converges in $C^{2+\lambda}(\overline{B(x_0, \frac{r_1}{2})})$. Since x_0 was arbitrary, it follows that $u \in C^{2+\alpha}(\mathbb{R}^n)$ and hence a solution to (1).

Case 2: $p(x) \in C(\Omega)$ or $q(x) \in C(\Omega)$. Since the sequence $\{u_k\}_1^\infty$ converges in $C^{1+\lambda}(\overline{B(x_0, r_1)})$ we have that $u_k \overset{s-C(B(x_0,r_1))}{\longrightarrow} u$, and consequently $\triangle u_k = p(x)u_k^\alpha$ $+q(x)u_k^\beta \overset{s-C(B(x_0,r_1))}{\longrightarrow} \triangle u = p(x)u^\alpha + q(x)u^\beta \equiv z$. Using the fact that the laplacian is a closed linear operator implies that $u \in D(\triangle)$, and $\triangle u = z$. Furthermore, since x_0 was chosen arbitrarily, we have that u is a classical solution of (1).

Now, all we must show is that our solution u is a large solution. We will prove that $u(x) \longrightarrow \infty$ as $x \longrightarrow \partial\Omega$ since $\{u_k\}$ is monotone with $u_k = k$ on $\partial\Omega$. To see this, let $x_0 \in \partial\Omega$ and let $\{x_j\}$ be a sequence in Ω such that $x_j \longrightarrow x_0$ as $j \longrightarrow \infty$. Let $k \in \mathbb{N}$. Since $\{u_k\}$ is monotone, choose $N_k \in \mathbb{N}$ such that $u_k(x_j) > k - 1$ for $j \geq N_k$. Thus, $u_m(x_j) > k - 1$ for $m \geq k$ and $j \geq N_k$. Therefore, given any $A > 0$, k and N_k can be chosen large so that $u(x_j) \geq A$ for $j \geq N_k$. Thus, $\lim_{j\to\infty} u(x_j) = \infty$, and hence, $\lim_{x\to x_0} u(x) = \infty$. Since x_0 was arbitrary, it is now apparent that u is a large solution of (1). ■

To prove our next result, we will need the help of the following lemma from [18].

Lemma 18 *(Lemma 2.0.18 of [18]) Let $x_0 \in \mathbb{R}^n \backslash \overline{\Omega}$, $n \geq 3$, and define $h(r) = (1+r^2)^{-\frac{1}{2}}$, where $r(x) \equiv |x - x_0|$. Then, $\triangle h(r) < 0$ on $\overline{\Omega}$.*

We now establish conditions on p and q for the existence of an entire large solution to (1). Previous work makes it clear that some restriction must be placed on the functions p and q if we expect (1) to have an entire large solution. So, as in some of the previous works on the single-term equation, we add an asymptotic condition to the functions p and q and prove the existence of a large entire solution of (1). The following theorem extends Theorem 2 of [11].

Theorem 19 *Suppose $p, q \in C(\mathbb{R}^n), n \geq 3,$ are nonnegative and c-positive. Then (1) has a nontrivial entire large positive solution if $0 < \alpha \leq \beta$, $\beta > 1$, and (7) and (10) hold.*

Proof. From Theorem 17, we have that for each $k \in \mathbb{N}$, there exists a positive solution to the boundary value problem

$$\begin{aligned} \triangle v_k = p(x)v_k^\alpha + q(x)v_k^\beta, \quad |x| < k, \\ v_k(x) \longrightarrow \infty \text{ as } |x| \longrightarrow k. \end{aligned} \tag{25}$$

Now, we know that for any k and $|x| \geq k$, $v_{k+1} \leq v_k = \infty$. Thus, it is apparent by the maximum principle (Theorem 3) that $v_1 \geq v_2 \geq ... \geq v_k \geq v_{k+1} \geq ... > 0$ in $\mathbb{R}^n$. In fact, suppose this is not true. That is, suppose for some k, $v_{k+1} > v_k$, for some x. Then, $\max\limits_{|x| \leq k}(v_{k+1} - v_k) > 0$. Let x_0 be the point where the maximum occurs. Notice that since $v_{k+1} \leq v_k = \infty$ for $|x| \geq k$, we know $|x_0| < k$. So, at x_0, we have

$$\begin{aligned} 0 &\geq \triangle(v_{k+1} - v_k) \\ &= \left(p(x)v_{k+1}^\alpha + q(x)v_{k+1}^\beta\right) - \left(p(x)v_k^\alpha + q(x)v_k^\beta\right) \\ &> \left(p(x)v_k^\alpha + q(x)v_k^\beta\right) - \left(p(x)v_k^\alpha + q(x)v_k^\beta\right) \\ &= 0, \end{aligned}$$

which is a contradiction. Therefore, $\{v_k\}$ is monotone. Thus, we need to prove that $\{v_k\}$ converges to some $v \in C(\mathbb{R}^n)$ and that $v \longrightarrow \infty$ as $|x| \longrightarrow \infty$.

To prove that $\{v_k\}$ converges we first note that the proof of Theorem 2 in [11] tells us that (7) and (10) imply

$$\int_0^\infty r^{1-n}\int_0^r s^{n-1}(\phi(s)+\psi(s))dsdr < \infty \tag{26}$$

Thus, $z(r) \equiv C + (1-\beta)\int_0^r t^{1-n}\int_0^t s^{n-1}(\phi(s)+\psi(s))ds\ dt$
where $C = (\beta-1)\int_0^\infty r^{1-n}\int_0^r s^{n-1}(\phi(s)+\psi(s))ds\ dr$ is the unique positive solution of $(r=|x|)$

$$\triangle z = (1-\beta)(\phi(r)+\psi(r)), \quad x \in \mathbb{R}^n, \tag{27}$$

$$z \longrightarrow 0 \text{ as } |x| \longrightarrow \infty.$$

We claim that $(v_k+1)^{1-\beta} \leq z$ on $|x| \leq k$. Clearly, when $|x| = k$, $(v_k+1)^{1-\beta} = 0$, and thus for $|x| = k$, $(v_k+1)^{1-\beta} \leq z$. Now, we will show that $(v_k+1)^{1-\beta} \leq z + \varepsilon(1+r^2)^{-\frac{1}{2}}$, $\forall \varepsilon > 0$, $|x| < k$. To do this, take $\varepsilon > 0$ and assume the inequality does not hold. Then, $\max\limits_{|x|\leq k}((v_k+1)^{1-\beta} - z - \varepsilon(1+r^2)^{-\frac{1}{2}}) > 0$. At the point where the maximum occurs, we have

$$\begin{aligned}
0 &\geq \triangle((v_k+1)^{1-\beta} - z - \varepsilon(1+r^2)^{-\frac{1}{2}}) \\
&= (1-\beta)(v_k+1)^{-\beta}\triangle v_k + (1-\beta)(-\beta)(v_k+1)^{-\beta-1}|\nabla v_k|^2 - \triangle z - \varepsilon\triangle(1+r^2)^{-\frac{1}{2}} \\
&= (1-\beta)(v_k+1)^{-\beta}[p(x)v_k^\alpha + q(x)v_k^\beta] + (1-\beta)(-\beta)(v_k+1)^{-\beta-1}|\nabla v_k|^2 \\
&\quad -(1-\beta)(\phi(r)+\psi(r)) - \varepsilon\triangle(1+r^2)^{-\frac{1}{2}} \\
&\geq (1-\beta)(v_k+1)^{-\beta}[p(x)(v_k+1)^\alpha + q(x)(v_k+1)^\beta] \\
&\quad +(1-\beta)(-\beta)(v_k+1)^{-\beta-1}|\nabla v_k|^2 - (1-\beta)(p(x)+q(x)) - \varepsilon\triangle(1+r^2)^{-\frac{1}{2}} \\
&\geq (1-\beta)(v_k+1)^{-\beta}(p(x)+q(x))(v_k+1)^\beta \\
&\quad +(\beta-1)(\beta)(v_k+1)^{-\beta-1}|\nabla v_k|^2 - (1-\beta)(p(x)+q(x)) - \varepsilon\triangle(1+r^2)^{-\frac{1}{2}} \\
&= (1-\beta)(p(x)+q(x)) + (\beta-1)(\beta)(v_k+1)^{-\beta-1}|\nabla v_k|^2 \\
&\quad -(1-\beta)(p(x)+q(x)) - \varepsilon\triangle(1+r^2)^{-\frac{1}{2}} \\
&= (\beta-1)(\beta)(v_k+1)^{-\beta-1}|\nabla v_k|^2 - \varepsilon\triangle(1+r^2)^{-\frac{1}{2}} \\
&\geq -\varepsilon\triangle(1+r^2)^{-\frac{1}{2}} > 0,
\end{aligned}$$

by Lemma 18, which gives a contradiction. Thus, $(v_k+1)^{1-\beta} \leq z + \varepsilon(1+r^2)^{-\frac{1}{2}}$, $\forall \varepsilon > 0$. So, $(v_k+1)^{1-\beta} \leq z$ if $|x| \leq k$. Let $w = z^{-(\beta-1)^{-1}} - 1$ and note that $v_k \geq w$ in $\mathbb{R}^n$ for all k. Therefore, the sequence $\{v_k\}$ is monotone and bounded. Thus, $\{v_k\}$ converges to some $v \in C(\mathbb{R}^n)$. Also, $v \geq w$ in $\mathbb{R}^n$. Since $w \longrightarrow \infty$ as $|x| \longrightarrow \infty$, $v \longrightarrow \infty$ as $|x| \longrightarrow \infty$. This concludes the proof. ∎

The biggest challenge in the previous proof involved being able to compare the terms $(v_k+1)^\alpha$ and $(v_k+1)^\beta$. The proof of Theorem 2 in [11] considered $(v_k)^{1-\beta}$ instead of $(v_k+1)^{1-\beta}$. Since it was possible that $\alpha < 1$, we needed to insure that the function we were considering was greater than one. By looking at $(v_k+1)^{1-\beta}$ we were able to compare $(v_k+1)^\alpha$ and $(v_k+1)^\beta$ and thus achieve the necessary contradiction.

2.2 Sublinear Case ($0 < \alpha \leq \beta \leq 1$)

We now move on to the sublinear case, where $0 < \alpha \leq \beta \leq 1$. For this case we have both existence and nonexistence results. Since the sublinear case is more complicated and fewer results have previously been discovered, our results for this case require more assumptions. Our existence results are limited to the radial case or to the existence of bounded solutions instead of large solutions.

2.2.1 Existence Results. Our first existence result extends Theorem 1 of [13] and provides a necessary and sufficient condition for the existence of an entire large solution for the radial case of (1). Before we present it, though, we establish the following lemma from [20] (See pg. 112), which will help us prove the result.

Lemma 20 *Let α and β be nonnegative real numbers, and suppose $0 < \lambda < 1$. Then $\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta$ with equality only if $\alpha = \beta$.*

Theorem 21 *Suppose $0 < \alpha \leq \beta < 1$ and suppose that $p(x) = p(|x|) \in C(\mathbb{R})$, and $q(x) = q(|x|) \in C(\mathbb{R})$ such that p and q are nonnegative. Then, the equation*

$$\triangle u = p(|x|)u^\alpha + q(|x|)u^\beta, \qquad x \in \mathbb{R}^n \tag{28}$$

has a large positive solution if and only if (8) or (11) holds.

Proof. To prove the necessity, we assume that (8) and (11) are not true. That is, assume

$$\int_0^\infty rp(r)dr < \infty, \text{ and } \int_0^\infty rq(r)dr < \infty. \tag{29}$$

We will show that (28) has no large positive solution. To do this, suppose that (28) does have a positive solution $u(x)$. Define

$$\overline{u}(r) \equiv \frac{1}{v_0(s^{n-1}r)} \int_{|x|=r} u(x)d\sigma_r \equiv \int_{|x|=r} u(x)d\sigma \tag{30}$$

where $v_0(s^{n-1}r)$ is the volume of the ball inside the $(n-1)$-dimensional sphere of radius r and σ_r is the measure on the sphere. We have

$$\begin{aligned}
\triangle\overline{u} &= \overline{u}'' + \frac{n-1}{r}\overline{u}' \\
&= \int_{|x|=r} \triangle u(x)d\sigma \\
&= p(r)\int_{|x|=r} u^\alpha d\sigma + q(r)\int_{|x|=r} u^\beta d\sigma \\
&\leq p(r)\left[\int_{|x|=r} ud\sigma\right]^\alpha + q(r)\left[\int_{|x|=r} ud\sigma\right]^\beta \\
&= p(r)\overline{u}^\alpha(r) + q(r)\overline{u}^\beta(r).
\end{aligned}$$

Thus we have

$$\overline{u}'' + \frac{n-1}{r}\overline{u}' \leq p(r)\overline{u}^\alpha(r) + q(r)\overline{u}^\beta(r). \tag{31}$$

Integrating the above inequality and using the fact that $\overline{u}' \geq 0$ yields

$$\begin{aligned}
\overline{u}(r) &\leq \overline{u}(r_0) + \int_{r_0}^r t^{1-n}\int_0^t s^{n-1}\left[p(s)\overline{u}^\alpha(s) + q(s)\overline{u}^\beta(s)\right] ds\ dt \\
&\leq \overline{u}(r_0) + \left(\overline{u}^\alpha + \overline{u}^\beta\right)\int_{r_0}^r t^{1-n}\int_0^t s^{n-1}\left[p(s)+q(s)\right] ds\ dt,
\end{aligned} \tag{32}$$

for $r \geq r_0 \geq 0$. Now, as in the proof of Theorem 19, notice that (29) implies $\int_0^\infty r^{1-n}\int_0^r s^{n-1}(p(s)+q(s))ds\ dr < \infty$. Since this is true, we can choose r_0 large so that

$$\gamma \equiv \int_{r_0}^\infty t^{1-n}\int_0^t s^{n-1}(p(s)+q(s))ds\ dr < \frac{1}{2}. \tag{33}$$

Since $0 < \alpha \leq \beta \leq 1$, we know $\overline{u}(r)^\alpha \leq 1+\overline{u}(r)$ and $\overline{u}(r)^\beta \leq 1+\overline{u}(r)$. Thus, $\overline{u}(r)^\alpha+\overline{u}(r)^\beta \leq 2+2\overline{u}(r)$. Hence, inequality (33) yields

$$\overline{u}(r) \leq \overline{u}(r_0) + \gamma(2+2\overline{u}(r)) \quad \forall r \geq r_0, \tag{34}$$

which yields

$$\begin{aligned} \overline{u}(r) - 2\gamma\overline{u}(r) &\leq \overline{u}(r_0) + 2\gamma \\ &\Rightarrow \overline{u}(r)(1-2\gamma) \leq \overline{u}(r_0) + 2\gamma \\ &\Rightarrow \overline{u}(r) \leq \left[\overline{u}(r_0) + 2\gamma\right](1-2\gamma)^{-1} \end{aligned} \tag{35}$$

Thus, $\overline{u}$ is bounded and therefore u cannot be a large solution. This completes the necessity part of the proof. To prove sufficiency we assume (8) or (11) is true, and will show that the equation

$$v''(r) + \frac{n-1}{r}v'(r) = p(r)v^\alpha(r) + q(r)v^\beta(r) \tag{36}$$

has a positive solution such that $v(r) \longrightarrow \infty$ as $r \longrightarrow \infty$. It suffices to show that for any fixed $c > 0$, the operator

$$T : C([0,\infty)) \longrightarrow C([0,\infty)),$$

defined by

$$Tu(r) = c + \int_0^r s^{1-n} \int_0^s t^{n-1}[p(t)u^\alpha(t) + q(t)u^\beta(t)]dt\ ds \tag{37}$$

has a fixed point in $C([0,\infty))$. In fact, assuming for the moment that such a fixed point u exists, we prove that $u(r) \longrightarrow \infty$ as $r \longrightarrow \infty$. This can be done by establishing that (8) and (11) each imply

$$\int_0^\infty s^{1-n} \int_0^s t^{n-1}(p(t)+q(t))dtds = \infty. \tag{38}$$

In fact the same analysis used in the proof of Theorem 1 in [13] proves that (8) and (11) each imply (38).

We now show that T has a fixed point in $C([0,\infty))$. To do this, we first establish a fixed point in $C([0,R))$ for any $R>0$. We consider the successive approximation, letting $u_0 = c$. Define $u_{k+1} = Tu_k$, $k = 0,1,2,\ldots,$. Notice that $c \le u_k$, $k = 0,1,2,\ldots,$ and $0 \le u_k'$. For the case $0<\alpha\le\beta<1$,

$$\begin{aligned}
u_{k+1}(r) &= c + \int_0^r s^{1-n}\int_0^s t^{n-1}[p(t)u_k^{\alpha}(t) + q(t)u_k^{\beta}(t)]dt\ ds \\
&\le c + \left[u_k^{\alpha}(r) + u_k^{\beta}(r)\right] H(r), \quad H(r) \equiv \int_0^r s^{1-n}\int_0^s t^{n-1}[p(t)+q(t)]dt\ ds \\
&= c + u_k^{\alpha}(r)H(r) + u_k^{\beta}(r)H(r) \\
&= c + u_k^{\alpha}(r)H^{\frac{1-\alpha}{1-\alpha}}(r) + u_k^{\beta}(r)H^{\frac{1-\beta}{1-\beta}}(r) \\
&\le c + \alpha u_k(r) + (1-\alpha)H^{\frac{1}{(1-\alpha)}}(r) + \beta u_k(r) + (1-\beta)H^{\frac{1}{(1-\beta)}}(r) \\
&= c + (\alpha+\beta)u_k(r) + (1-\alpha)H^{\frac{1}{(1-\alpha)}}(r) + (1-\beta)H^{\frac{1}{(1-\beta)}}(r),
\end{aligned} \tag{39}$$

where we can make the last step by applying the previous lemma. We will now use the Principle of Mathematical Induction to prove that

$$u_k(r) \le \frac{c}{2(1-\alpha)} + \frac{c}{2(1-\beta)} + H^{\frac{1}{(1-\alpha)}}(r) + H^{\frac{1}{(1-\beta)}}(r) \equiv M_r,\ \forall k. \tag{40}$$

Clearly, when $k=0$,

$$\begin{aligned}
u_0 &= c = \frac{(1-\alpha)c}{2(1-\alpha)} + \frac{(1-\beta)c}{2(1-\beta)} \\
&\le \frac{c}{2(1-\alpha)} + \frac{c}{2(1-\beta)} \\
&\le \frac{c}{2(1-\alpha)} + \frac{c}{2(1-\beta)} + H^{\frac{1}{(1-\alpha)}}(r) + H^{\frac{1}{(1-\beta)}}(r).
\end{aligned}$$

Now, let (40) be true for some k. We will show that (40) is true for $k+1$. From (39) we know that

$$\begin{aligned}
u_{k+1}(r) &\leq c+(\alpha+\beta)u_k(r)+(1-\alpha)H^{\frac{1}{(1-\alpha)}}(r)+(1-\beta)H^{\frac{1}{(1-\beta)}}(r) \\
&\leq c+(\alpha+\beta)\left[\frac{c}{2(1-\alpha)}+\frac{c}{2(1-\beta)}+H^{\frac{1}{(1-\alpha)}}(r)+H^{\frac{1}{(1-\beta)}}(r)\right] \\
&\quad +(1-\alpha)H^{\frac{1}{(1-\alpha)}}(r)+(1-\beta)H^{\frac{1}{(1-\beta)}}(r) \\
&= \frac{(1-\alpha)c}{2(1-\alpha)}+\frac{(1-\beta)c}{2(1-\beta)}+\frac{\alpha c}{2(1-\alpha)}+\frac{\beta c}{2(1-\beta)}+\alpha H^{\frac{1}{(1-\alpha)}}(r) \\
&\quad +\beta H^{\frac{1}{(1-\beta)}}(r)+(1-\alpha)H^{\frac{1}{(1-\alpha)}}(r)+(1-\beta)H^{\frac{1}{(1-\beta)}}(r) \\
&= \frac{c}{2(1-\alpha)}+\frac{c}{2(1-\beta)}+H^{\frac{1}{(1-\alpha)}}(r)+H^{\frac{1}{(1-\beta)}}(r).
\end{aligned}$$

So, by the Principle of Mathematical Induction (40) is true for all k. Thus $c \leq u_k(r) \leq M_R$, $r \in [0,R]$. Furthermore, u_k' is bounded since

$$\begin{aligned}
u_k'(r) &= r^{1-n}\int_0^r t^{n-1}[p(t)u_{k-1}^{\alpha}(t)+q(t)u_{k-1}^{\beta}(t)]dt \\
&\leq M_R^{\alpha}r^{1-n}\int_0^r t^{n-1}p(t)dt+M_R^{\beta}r^{1-n}\int_0^r t^{n-1}q(t)dt \\
&\leq M_R\left[\int_0^R p(t)dt+\int_0^R q(t)dt\right]
\end{aligned}$$

and $u_k' \geq 0$. Thus for $0<\alpha\leq\beta<1$, the sequence $\{u_k\}$ is bounded and equicontinuous on $[0,R]$. By the Arzela-Ascoli Theorem (Theorem 5), $\{u_k\}$ has a uniformly convergent subsequence on $[0,R]$.

Assuming then that $u_{k_j} \longrightarrow u$ on $[0,R]$, it is clear that $u \in C([0,R])$ and $Tu=u$ on $[0,R]$. To prove that T has a fixed point in $C([0,\infty))$, we let $\{w_k\}$ be defined as follows:

$$Tw_k = w_k \quad \text{on} \quad [0,k],\ w_k \in C([0,k]). \tag{41}$$

As we did previously in this proof, it can be shown that $\{w_k\}$ is bounded and equicontinuous on $[0,1]$. Thus, $\{w_k\}$ has a subsequence, $\{w_k^1\}$, which converges uniformly on $[0,1]$. Let

$$w_k^1 \longrightarrow v_1 \qquad \text{on} \quad [0,1] \text{ as } k \longrightarrow \infty.$$

Likewise, the subsequence $\{w_k^1\}$ is bounded and equicontinuous on $[0,2]$ so that it has a subsequence $\{w_k^2\}$ which converges uniformly on $[0,2]$. Let

$$w_k^2 \longrightarrow v_2 \qquad \text{on } [0,2] \text{ as } k \longrightarrow \infty.$$

Note that $w_k^2 \longrightarrow v_1$ on $[0,1]$ since $\{w_k^2\}$ is a subsequence of $\{w_k^1\}$. Thus $v_2 = v_1$ on $[0,1]$. Continuing this line of reasoning, we obtain a sequence $\{v_k\}$ with the following properties:

$$\begin{aligned} v_k &\in C([0,k]), \qquad k = 1, 2, \ldots, \\ v_k(r) &= v_1(r), \qquad \forall r \in [0,1], \\ v_k(r) &= v_2(r), \qquad \forall r \in [0,2], \\ &\vdots \\ v_k(r) &= v_{k-1}(r), \qquad \forall r \in [0,k-1]. \end{aligned}$$

Therefore, it is clear that $\{v_k\}$ converges to v where

$$v(r) = v_k(r) \quad \text{if } 0 \leq r \leq k \tag{42}$$

and the convergence is uniform on bounded sets. Hence $v \in C([0,\infty])$ and satisfies $Tv = v$ if $0 < \alpha \leq \beta < 1$.
Therefore, T has a fixed point in $C([0,\infty))$ for $0 < \alpha \leq \beta < 1$. This completes the proof.
∎

One of the challenges of the previous proof was modifying the bounds used in the proof of Theorem 1 of [13] to work for the multi-term equation. Since multiple terms had to be considered, the bounds did not directly follow from the proof of the earlier result.

We now establish conditions for the existence of an entire bounded solution of (1) in $\mathbb{R}^n$, where p and q are locally Hölder continuous in $\mathbb{R}^n$. This result extends the existence part of Theorem 3 of [13].

Theorem 22 *Suppose p and q are nonnegative and locally Hölder continuous in $\mathbb{R}^n$ and (7) and (10) hold. Then, (1) has a nonnegative nontrivial entire bounded solution in $\mathbb{R}^n$ if* $0 < \alpha \leq \beta < 1$.

Proof. Define the function $\theta(r)$ by $\theta(r) = \max_{|x|=r}\{p(x),\ q(x)\}$. Now, consider the function z defined by

$$z(r) = 1 + \int_0^r s^{1-n} \int_0^s t^{n-1}\theta(t)[z^\alpha(t) + z^\beta(t)]dt\ ds \tag{43}$$

We will show that z is a bounded solution to $\triangle z = \theta(z^\alpha + z^\beta)$. Note that $\triangle z = \theta(z^\alpha + z^\beta) \geq pz^\alpha + qz^\beta$. We now show that z is bounded. Let $z_0 = 1$, and define z_k, $k = 1, 2, \ldots$, by

$$z_k = 1 + \int_0^r s^{1-n} \int_0^s t^{n-1}\theta(t)[z_{k-1}^\alpha(t) + z_{k-1}^\beta(t)]dt\ ds. \tag{44}$$

We will now use induction to prove that the sequence $\{z_k\}$ is increasing. When $k = 0$, clearly

$$\begin{aligned} z_0 &= 1 \leq z_1 = 1 + \int_0^r s^{1-n} \int_0^s t^{n-1}\theta(t)[z_0^\alpha(t) + z_0^\beta(t)]dt\ ds \\ &= 1 + \int_0^r s^{1-n} \int_0^s 2t^{n-1}\theta(t)dt\ ds \end{aligned}$$

Now, suppose $z_k \leq z_{k+1}$ for some k. We will show that $z_{k+1} \leq z_{k+2}$. We know that

$$\begin{aligned} z_{k+1} &= 1 + \int_0^r s^{1-n} \int_0^s t^{n-1}\theta(t)[z_k^\alpha(t) + z_k^\beta(t)]dt\ ds \\ &\leq 1 + \int_0^r s^{1-n} \int_0^s t^{n-1}\theta(t)[z_{k+1}^\alpha(t) + z_{k+1}^\beta(t)]dt\ ds \\ &= z_{k+2} \end{aligned}$$

Thus, by induction the sequence $\{z_k\}$ is increasing. Now, since $z_k' > 0$, $z_k(r)$ is increasing for all k. Also, notice that $z_k \geq 1$ for all k. Thus, for $k > 1$,

$$\begin{aligned}
z_k &= 1 + \int_0^r s^{1-n} \int_0^s t^{n-1}\theta(t)[z_{k-1}^\alpha(t) + z_{k-1}^\beta(t)]dt\ ds \\
&\leq 1 + \int_0^r s^{1-n} \int_0^s t^{n-1}\theta(t)[z_k^\alpha(t) + z_k^\beta(t)]dt\ ds \\
&\leq 1 + [z_k^\alpha(r) + z_k^\beta(r)] \int_0^r s^{1-n} \int_0^s t^{n-1}\theta(t)dt\ ds \\
& \quad\ 1 + [z_k^\alpha(r) + z_k^\beta(r)] \int_0^\infty s^{1-n} \int_0^s t^{n-1}\theta(t)dt\ ds \\
&\leq 1 + [z_k^\alpha(r) + z_k^\beta(r)]M \\
&\leq z_k^\beta(r) + [z_k^\beta(r) + z_k^\beta(r)]M \\
&= z_k^\beta(r) + 2z_k^\beta(r)M, \qquad (45)
\end{aligned}$$

where $M = \displaystyle\int_0^\infty s^{1-n} \int_0^s t^{n-1}\theta(t)dt\ ds$. As in the proof of Theorem 20, we know that (7) and (10) imply that $M < \infty$. Now, (45) implies

$$\begin{aligned}
z_k^{1-\beta} &\leq 1 + 2M \\
&\Rightarrow\ z_k \leq (1+2M)^{(1-\beta)^{-1}} \equiv M_0 \qquad (46)
\end{aligned}$$

Thus, $\{z_k\}$ is uniformly bounded by M_0. Since the sequence $\{z_k\}$ is monotone and bounded, we know that the limit of $\{z_k\}$ exists. We now let $\lim_{k\to\infty} z_k = z$, and we will show that z is, in fact, the function we defined in (43). Since z_k is integrable for all k and z is integrable, we know by Theorem 9.12 of [2] that,

$$\begin{aligned}
z &= \lim_{k\to\infty} z_k = \lim_{k\to\infty}\left[1 + \int_0^r s^{1-n} \int_0^s t^{n-1}\theta(t)[z_{k-1}^\alpha(t) + z_{k-1}^\beta(t)]dt\ ds\right] \\
&= 1 + \int_0^r s^{1-n} \int_0^s t^{n-1}\theta(t) \lim_{k\to\infty}[z_{k-1}^\alpha(t) + z_{k-1}^\beta(t)]dt\ ds \\
&= 1 + \int_0^r s^{1-n} \int_0^s t^{n-1}\theta(t)[z^\alpha(t) + z^\beta(t)]dt\ ds
\end{aligned}$$

Therefore, $z \leq M_0$. The standard bootstrap argument that was applied in the proof of Theorem 17 can now be used to show that z is, in fact, a bounded solution to $\triangle z = \theta(z^\alpha + z^\beta)$. Let $\underline{u} = z$. Now, let $\overline{u} = M_0$. Clearly, $\triangle \overline{u} = 0 \leq p\overline{u}^\alpha + q\overline{u}^\beta$, and $\underline{u} \leq \overline{u}$

by definition. Thus, we know that there exists a nonnegative nontrivial entire bounded solution for (1) by the upper/lower solution method (Theorem 2). This completes the proof. ∎

The proof of the previous result did not follow at all from the proof of Theorem 3 of [13]. We needed to construct the upper solution we used, which led directly to lower solution that we utilized.

2.2.2 Nonexistence Results. Our final two results are nonexistence results. The first is an extension of Theorem 2 of [13] and establishes that for the sublinear case, (1) has no positive large solution for a bounded domain Ω.

Theorem 23 *Suppose Ω is a bounded domain in $\mathbb{R}^n, n \geq 2$, and p and q are continuous on $\overline{\Omega}$. If $0 < \alpha \leq \beta \leq 1$, then (1) has no positive, large solution in Ω.*

Proof. Suppose u is such a solution. Let $v(x) = \log(1 + u(x))$. Then,

$$\begin{aligned}
\triangle v &= -(1+u)^{-2}|\nabla u|^2 + (1+u)^{-1}(p(x)u^\alpha + q(x)u^\beta) \\
&\leq (1+u)^{-1}u^\alpha p(x) + (1+u)^{-1}u^\beta q(x) \\
&\leq p(x) + q(x) \\
&\leq K
\end{aligned}$$

for some constant $K > 0$ since p and q are continuous on $\overline{\Omega}$, a compact set. Thus, we have $\triangle(v - K|x|^2) < 0$, $x \in \Omega$.

The proof is now identical to that of Theorem 2 in [13]. ∎

Our final result is closely related to Theorem 22. It provides conditions under which (1) does not have an entire bounded solution in $\mathbb{R}^n$ and is an extension of the nonexistence part of Theorem 3 of [13].

Theorem 24 *Suppose p and q are locally Hölder continuous in $\mathbb{R}^n$. If (9) holds or (12) holds then (1) has no nonnegative bounded entire solution in $\mathbb{R}^n$, $n \geq 3$, for $0 < \alpha \leq \beta \leq 1$.*

Proof. Let (9) be true and suppose such a solution exists. That is, suppose there exists u, nonnegative and bounded in $\mathbb{R}^n$, $n \geq 3$ such that $\triangle u = p(x)u^\alpha + q(x)u^\beta$. Now, consider the equation

$$\triangle v = p(x)v^\alpha. \tag{47}$$

Note that $\triangle u = p(x)u^\alpha + q(x)u^\beta \geq p(x)u^\alpha$. Let $\underline{v} = u$. Now, let $\overline{v} = M = \sup u$. We can make this definition because we know u is bounded. Then, $\triangle \overline{v} = \triangle M = 0 \leq p(x)\overline{v}^\alpha$. Thus, by the upper/lower solution method, there exists a nontrivial, nonnegative, entire bounded solution to (50). But, this contradicts Theorem 3 of [13], which says (50) has no such solution. Therefore, u must not exist.
Similarly, let (12) be true and suppose such a solution exists. Now, consider the equation

$$\triangle v = q(x)v^\beta. \tag{48}$$

Note that $\triangle u = p(x)u^\alpha + q(x)u^\beta \geq q(x)u^\alpha$. Let $\underline{v} = u$. Now, let $\overline{v} = M = \sup u$. We can make this definition because we know u is bounded. Then, $\triangle \overline{v} = \triangle M = 0 \leq q(x)\overline{v}^\beta$. Thus, by the upper/lower solution method, there exists a nontrivial, nonnegative, entire bounded solution to (51). But, this contradicts Theorem 3 of [13], which says (51) has no such solution. Therefore, u must not exist. This completes the proof. ■

As with the proof of Theorem 21, the proof of the previous result did not follow from the proof of Theorem 3 of [12]. Still, clearly Theorem 3 of [12] did enable us to find the necessary contradtictions.

III. Conclusion

3.1 Conclusion

We began our research in search of conditions for the existence of large solutions to the semilinear elliptic equation

$$\triangle u = p(x)u^{\alpha} + q(x)u^{\beta}, \qquad x \in \Omega \subseteq \mathbb{R}^n,\ n \geq 3. \tag{49}$$

Our results fell into two cases, the superlinear/mixed ($0 < \alpha \leq \beta, \beta > 1$) case and the sublinear ($0 < \alpha \leq \beta \leq 1$) case. The multi-term equation had previously only been considered by Lair and Wood [11], whose results were included as a special case of our results. There had been significant study into the single-term equation

$$\triangle u = p(x)u^{\gamma}. \tag{50}$$

Our first result established conditions for the existence of solutions to Eq. (1) on a bounded domain $\Omega \subseteq \mathbb{R}^n$ in the superlinear/mixed case. We used the upper/lower solution method to create a bounded and monotone sequence $\{u_k\}$, which consequently converged to a function u. We then used the standard bootstrap argument to show that u was actually a solution to (1). Then, we considered conditions for the existence of entire large solutions to Eq. (1). We extended the results of [12], showing that if (7) and (10) hold then there exists an entire large solution for (1).

Next, we looked at the sublinear case. To our knowledge, there are few results for this case, even for the single-term equation. We extended the results of [13] to establish two existence results and two nonexistence results. Our first existence result was for the radial case, where $p(x) = p(|x|)$ and $q(x) = q(|x|)$. We showed that (8) or (11) were necessary and sufficient conditions for the existence of an entire large solution for the radial case of (1) for the sublinear problem. We also established a condition for the existence of an entire bounded solution in $\mathbb{R}^n$. We showed that if (7) and (10) hold, and p and q are Hölder continuous, then (1) has an entire bounded solution in $\mathbb{R}^n$.

Finally, we established some nonexistence results for the sublinear case. We first showed that, for a bounded domain Ω, (1) has no large solutions on Ω. In addition, we established that if (9) or (12) holds, and p and q are Hölder continuous, then there does not exist a bounded large solution for (1) in the sublinear case.

In this work we have laid significant groundwork for further study of problems of this type. We extended many of the more recent results for the single-term semilinear elliptic equation to the multi-term semilinear elliptic equation. It is our sincere hope that our work will be helpful to others who are studying this problem now and those who will study this problem in the future.

3.2 Further Work

Though we were able to achieve several useful results, there is much work left to do in the study of this problem. First, we need to consider Theorems 22 and 23 for the case where $\beta = 1$. We are confident that both results hold for this case, but have not yet developed a proof for either result. Second, the multi-term equation offers the opportunity to put different conditions on the individual functions p and q. For example, it is left as an open problem whether there exist entire large solutions to (1) in the superlinear or mixed cases when either (7) or (10) does not hold, that is when either $\int_0^\infty r\phi(r)dr = \infty$ or $\int_0^\infty r\psi(r)dr = \infty$. It is our conjecture that under those conditions, the problem will behave much like it would if both (8) and (11) do not hold $\left(\text{i.e. } \int_0^\infty r\phi(r)dr = \infty \text{ and } \int_0^\infty r\psi(r)dr = \infty\right)$, meaning that we think that Eq. (1) does not have an entire large solution under those conditions. But, as of yet, our conjecture has not been proven.

There are also two other very interesting ways that our results could be extended. One area of additional study would be to examine existence of solutions in systems of multi-term equations. Also, Eq. (1) could be expanded to include any countable number of terms, opening up the options for many different combinations of conditions on the involved functions. The equation would probably behave similarly and have solutions under similar conditions, but it would still be interesting to see if there are any changes as the number of terms is increased. Overall, this area of study is wide open and offers the opportunity for valuable research in a range of problems.

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Vita

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REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 074-0188*

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED** *(From – To)*
01-09-2006	**Master's Thesis**	Sep 2005 – Sep 2006

4. TITLE AND SUBTITLE

Existence of Large Solutions to Semilinear Elliptic Equations with Multiple Terms

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)

Smith, David N., Second Lieutenant, USAF

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(S)

Air Force Institute of Technology
Graduate School of Engineering and Management (AFIT/EN)
2950 Hobson Way
WPAFB OH 45433-7765

8. PERFORMING ORGANIZATION REPORT NUMBER

AFIT/GAM/ENC/06-05

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

N/A

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

We consider the semilinear elliptic equation $\triangle u=p(x)u^{\alpha}+q(x)u^{\beta}$ on a domain $\Omega\subseteq R^n$, $n\geq 3$, where p and q are nonnegative continuous functions with the property that each of their zeroes is contained in a bounded domain Ω_{p} or Ω_{q}, respectively in Ω such that p is positive on the boundary of Ω_{p} and q is positive on the boundary of Ω_{q}. For Ω bounded, we show that there exists a nonnegative solution u such that $u(x)\to\infty$ as $x\to\partial\Omega$ if $0<\alpha\leq\beta$, $\beta>1$, and that such a solution does not exist if $0<\alpha\leq\beta\leq 1$. For $\Omega= R^n$, we establish conditions on p and q to guarantee the existence of a nonnegative solution u satisfying $u(x)\to\infty$ as $|x|\to\infty$ for $0<\alpha\leq\beta$, $\beta>1$, and for $0<\alpha\leq\beta\leq 1$. For $\Omega=R^n$ and $0<\alpha\leq\beta<1$, we also establish conditions on p and q for the existence and nonexistence of a solution u where u is bounded on R^n.

15. SUBJECT TERMS

Mathematical Analysis, Elliptic Differential Equations, Differential Equations, Large Solutions, Mathematics

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
					Dr. Aihua W. Wood (ENC)
REPORT	**ABSTRACT**	**c. THIS PAGE**	**UU**	41	**19b. TELEPHONE NUMBER** *(Include area code)*
U	**U**	**U**			(937) 255-3098, ext 4521; e-mail: Aihua.Wood@afit.edu