3-2020

Analysis of Beta Distribution for Subjective Uncertainty Analysis in Cost Models

Ryan D. Stafford

ANALYSIS OF BETA DISTRIBUTION FOR SUBJECTIVE UNCERTAINTY ANALYSIS IN COST MODELS

THESIS

Ryan D. Stafford, 1st Lieutenant, USAF

AFIT-ENV-20-M-241

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

The views expressed in this thesis are those of the author and do not reflect the official policy or position of the United States Air Force, the Department of Defense, or the United States Government.

AFIT-ENV-20-M-241

ANALYSIS OF BETA DISTRIBUTION FOR SUBJECTIVE UNCERTAINTY ANALYSIS IN COST MODELS

THESIS

Presented to the Faculty

Department of Systems Engineering and Management

Graduate School of Engineering and Management

Air Force Institute of Technology

Air University

Air Education and Training Command

In Partial Fulfillment of the Requirements for the

Degree of Master of Science in Cost Analysis

Ryan D. Stafford, BS

1st Lieutenant, USAF

March 2020

AFIT-ENV-MS-20-M-241

ANALYSIS OF BETA DISTRIBUTION FOR SUBJECTIVE UNCERTAINTY ANALYSIS IN COST MODELS

Ryan D. Stafford, BS

1st Lieutenant, USAF

Committee Membership:

Lt Col Clay M. Koschnick, Ph.D
Chair

David K. Fass, Ph.D
Member

Jonathan D. Ritschel, Ph.D
Member

John J. Elshaw, Ph.D
Member

Abstract

Subjective uncertainty exists within the realm of cost estimation. Typical methodology for subjective uncertainty involves elicitation from a subject matter expert to provide a high, low, and most likely value -- defining a triangular distribution -- to model said uncertainty. This manuscript explores ways to leverage research on elicitation geared towards defining a triangular distribution and provide a simple conversion to a beta distribution usable by cost analysts with various degrees of mathematical knowledge. Furthermore, this manuscript attempts to demonstrate the benefits of using a beta distribution through its application as a conjugate prior for Bayesian updating in cost models.

Table of Contents

List of Figures

List of Tables

Introduction of Articles

The articles included serve to provide a complete methodology to take a given triangular distribution and convert to a beta distribution, and use the beta distribution for Bayesian updating in a cost estimating framework. The Joint Agency Cost Schedule Risk and Uncertainty Handbook (JACSRUH) recommends using a triangular distribution for subjective uncertainty in cost models on the basis of the distribution's simplicity. While it's true the triangular distribution's parameters are considered more intuitive than that of the beta distribution, use of a beta distribution allows for alternative mathematical models to quantify uncertainty. Specifically, this thesis looks at using the beta distribution as a conjugate prior distribution for Bayesian updating.

The first article provides a methodology for converting a triangular distribution to a beta distribution. This analysis considers simplicity to be the top priority in developing methodology and for this reason chooses to primarily analyze a method of moments technique to convert from a triangular distribution to a beta distribution. The premise of the manuscript centers around the fact that an entry level cost-analyst with a fundamental understanding of probability, statistics, and algebra will be able to execute a method of moments conversion without any advanced knowledge on the aforementioned subjects. The manuscript compares the results of the method of moments for estimating the parameters of a beta distribution to quantile estimation which considers a more complete view of the distribution's shape in order to determine the validity of the results. Finally, the manuscript presents an empirical example to characterize the aggregate effect of converting to a beta distribution.

The second article focuses on using the beta distribution as a conjugate prior for Bayesian updating. Specifically, it looks at two applications: 1) combining a distribution defined through expert opinion elicitation with realized costs in order to update budget requests; 2) combining multiple experts' opinions into a single distribution for analysis. The paper reviews existing research on using Bayesian updating in the cost estimation field and highlights the uniqueness of this research. This manuscript uses a purely theoretical approach and will make suggestions on future research involving empirical analysis.

I. Introduction

Subjective uncertainty is a type of uncertainty where data does not exist in order to model the uncertainty and instead must be assessed in a biased manner. Brownstein et al. (2019) note that subjective uncertainty exists in all stages of scientific inquiry but that objective uncertainty -- which can be modeled with data -- is the fundamental goal. Research has been conducted to determine the best practices for elicitation of information from subject matter experts (SMEs) in order to reach this objective threshold. O'Hagan (2019) conducted research to discuss pitfalls such as anchoring and cognitive bias as well setting out best practices to make the elicitation process more rigorous. Burgman (2015) discussed ways to identify risky advice as well as the advantages of group estimates over individual estimates. Grafton and Selwyn (2012) conducted a South African case study to find ways to elicit tacit knowledge from experts through storytelling. Beyond simple elicitation of SME inputs there exists studies on mathematical representation of subjective uncertainty such as Machina and Schiedler (1992) modeling choice theory and Helton (1997) estimating the maintenance costs for the Turkish Air Force. Finally, Clemen and Reilly (2014) look at specific implementation of modeling subjective uncertainty in order to model costs while Jorgensen (2004) has reviewed studies of software estimation.

Uncertainty analysis in the cost estimation field provides decision makers a range of potential program costs. The Joint Agency Cost Schedule Risk and Uncertainty Handbook (JACSRUH) describes best practices for modeling uncertainty within the DoD. The handbook distinguishes between objective uncertainty which is based on the application of statistical processes and subjective uncertainty where the aforementioned processes cannot be used (Thomas & Fitch, 2014). This manuscript seeks to leverage previous works on SME elicitation and modeling as well as JACSRUH guidance as a foundation for justifying the use of a beta distribution in cost modeling. The goal is to keep the methodology for elicitation consistent while providing a simple method to cost analysts to convert from a triangular distribution to a beta distribution.

When eliciting expert opinion cost estimators seek to gather measures of central tendency (the center value of a probability distribution) and dispersion (how spread out a distribution is) in order to build distributions that capture the potential cost. The JACSRUH states that the typical dispersion parameters for subjective uncertainty are the minimum and maximum. There are four distributions defined by a minimum and maximum the JACSRUH considers for subjective uncertainty: uniform, betaPERT, triangular and beta. Note the standard beta distribution is not defined by a minimum and maximum, it is a normalized version of the four-parameter beta distribution which is defined by these additional parameters. Empirically, the uniform distribution was never found to be the best fit for representing costs (Smith et al., 2010). Furthermore, the practical application of this distribution is limited and represents a situation where no information is known about the level of uncertainty beyond the minimum and maximum. The betaPERT is reserved for cases with considerable knowledge of the mode such as cases where empirical data informs the SME. Additionally, the JASCRUH suggests using the betaPERT when a distribution is known to be left skewed so it may not be applicable for all scenarios. This leaves the triangular and beta for cases where there is a good idea of the mode and where there is little knowledge of the mode, respectively. Differentiating between the two, Clemen and Reilly (2014) suggest the triangular distribution is a good middle ground between the distribution with the best theoretical fit and an easily assessed distribution. Furthering this point, the JACSRUH shows the beta distribution to be the best fit in 19% of instances and the triangular the best in 18% of instances (Smith et al., 2010); since they are similar in frequency of best fit, the JACSRUH recommends using triangular distribution due to its simplicity. Additionally, Berny (1989) found in discussions with project managers that the mode a core piece of any estimate -- a parameter which can be directly stated by the SME and is a defining parameter of the triangular distribution.

Furthering the idea of simplicity, work has been done to convert a beta distribution to a triangular distribution due to its ease of use and more intuitive interpretability (Johnson, 1997). Additionally, Williams (1992) supports the use of the triangular distribution claiming the beta distribution is not easily understood and that its parameters are not easy to estimate. He conversely claims the triangular distribution is easier to understand and more comprehensible to the project planner. This manuscript aims to alleviate the issue of the simplicity argument of the triangular distribution over the beta distribution by finding an easy application to convert a given triangular distribution to a beta distribution. By using a beta distribution in place of a triangular distribution, analysts gain the advantage of using a distribution that can be used in other mathematical frameworks (e.g. as a conjugate prior for Bayesian updating) during the lifecycle of the program while maintaining a distribution fit empirically similar to that of triangular distribution.

II. Methodology

The triangular distribution is defined by three parameters: minimum (a), maximum (b), and mode (c). These parameters have an intuitive interpretation that allows a subject matter expert to provide a specific numerical estimate of each parameter. Conversely, the beta distribution is defined by two shape parameters: an alpha parameter (α) and a beta parameter (β). These parameters do not have the same intuition as those of the triangular distribution's parameters; although there are rules for these parameters that provide guidance for their impact on the overall shape of the distribution -- e.g., when both α and β are greater than one the resulting distribution is unimodal (i.e. contains one value that is the most likely to occur. Alternatively, this corresponds to one peak when viewing the distribution graphically). Additionally, there exists a four-parameter beta distribution that is defined between a minimum and maximum value (thus the extra two parameters). The more common two-parameter beta distribution -- which is defined over the support of [0,1] -- is just a standardized version of the four-parameter beta distribution, meaning its minimum and maximum values have been normalized to solely span the interval between 0 and 1. The triangular distribution has a minimum (a) and maximum (b) and is defined over the support [a,b]; it too can be standardized to have a support of [0,1] such that it is just a scaled version of the original triangular distribution. For purposes of this analysis, the standardized triangular and beta distributions are used under the reasoning that a triangular distribution obtained through elicitation can easily be standardized to the interval [0,1] by scaling the distribution such that the minimum (a) is 0, the maximum (b) is 1, and the mode (c) is a value between 0 and 1 that keeps the skewness of the original distribution intact. This standardization allows for comparisons to be more intuitively interpreted as percentages rather than discussing percentages for a distribution with a minimum and maximum values other than 0 and 1, respectively.

The goal of this study is to approximate a given triangular distribution with a beta distribution. Therefore, the triangular distribution is considered a known entity. From this triangular distribution, the defining parameters of a beta distribution (i.e., α and β) are estimated in order to achieve the best approximation of the given triangular distribution. In considering methods to estimate the parameters α and β, simplicity is considered an important factor. Analysts have a wide variety of mathematical knowledge and comfortability with statistical programming; therefore, a simplistic method of conversion is considered desirable in order to limit any form of additional training an analyst would need to convert from triangular to beta. Specifically, this manuscript uses method of moments based on the population defined by the known triangular distribution and compares the results to quantile estimation in order to characterize the fit.

Method of Moments

The first method considers different method of moments combinations to estimate the α and β parameters for a beta distribution. In this study, the word moments include both moments (such as mean and variance) and other common metrics like mode. Traditionally, a method of moments sets the population moment formula equal to the numerical value of a sample moment. The population moments for a beta distribution are a function of α and β (i.e. the numerical value for the mean of a beta distribution can be calculated knowing parameters α and β). Likewise, the population moments for a triangular distribution are a function of a, b, and c. Since a, b, and c are known through the elicitation process, in this research the population measures of the beta distribution are set equal to same measures from the given triangular distribution. Because the population measure of a beta distribution is a function of α and β only two moments from the triangular distribution are needed to fully define the beta distribution. This research considers five measures -- mean, median, mode, variance, and skewness -- for two distinct reasons. First, they are commonly used measures of central tendency and dispersion. Second, they are measures which cost analysts are assumed to be familiar with -- and therefore would not require advanced training in order to implement.

Given these five common measures there exist ten possible combinations to calculate the parameters α and β. Since the goal of this manuscript is to approximate a triangular distribution with a beta, only unimodal solutions are permitted (i.e. values of α and β to be estimated are restricted to being ≥ 1). This restriction also allows the median to be estimated algebraically rather than solving with the incomplete beta function (Kerman, 2011). The combinations of mean-variance, mean-mode, mean-median, median-mode, variance-skewness, mean-median, and mode-median all yield singular solutions under the restrictions placed on α and β. The combinations of median-skewness and mean-skewness yield no solutions under these restrictions leading to their exclusion from this analysis. The combination of mode-skewness results in only large values of α and β -- values greater than 100. Given Equation 1 for variance of a beta distribution

$$var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}, \tag{1}$$

subject to the constraints of α,β ≥ 100, is maximized at α=100 and β=100 with a variance of .0012, as values of α and β increase variance would decrease because the denominator is increasing at a rate faster than the numerator. The triangular distributions used for this analysis have variance ranging from .0417 to .0555. Since values of α and β are strictly greater than 100, the closest variance the mode-skewness combination can result in is 34.75 times smaller than the given triangular distributions. Given the magnitude of difference between the variances for the beta distribution and triangular distribution using the mode-skewness combination, it is also excluded from this analysis.

Quantile Estimation

The second method for estimating parameters α and β is quantile estimation. Quantile estimation is used to characterize the fit of the method of moments methodology because it estimates the response variables -- α and β -- without restricting attention to the conditional mean and variance (Davino et al., 2014). Since the beta distribution is not defined by its mean and variance, an estimation technique that does not restrict estimation to these parameters can provide a better empirical fit. Furthermore, using deciles within the quantile estimation framework allows α and β to be estimated using 11 data points from the original triangular distribution rather than the 2 used for method of moments. If the results from the method of moments follow closely to the quantile estimation, method of moments can be considered to be an adequate method for converting from triangular to beta.

For each triangular distribution, the PDF value for each decile is calculated. Then using these 11 paired values and the @risk software package, parameters α and β are estimated using a least-squares method of root-mean squared error minimizing the generic Equation 2

$$RMSErr = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(f(x_i,\alpha,\beta) - y_i)^2} \tag{2}$$

where α and β are the parameters of the theoretical distribution function, in this case a beta distribution, that minimizes the distance between the curve and the data points.

III. Results

For each methodology, 26 estimated beta distributions are compared to their corresponding triangular distributions for all combinations of measurements. The 26 distributions correspond to 26 triangular distributions (given) fixed to the support [0,1] with the mode varying along the same support at .04 intervals. This manuscript considers the skewness of these 26 distributions adequately diverse to draw conclusions about the results. To analyze similarities, the cumulative distribution functions (CDFs) of both triangular and beta are compared at 101 individual points equally spanning the bounds of the distributions [0,1] in order to account for differences at each percentile of the CDF. Absolute deviation (AD) is reported as its values are considered easier to interpret than mean-squared error (MSE) while providing a more comprehensive characterization than maximum absolute deviation (MAD). Note, analysis was also conducted using MSE and MAD and the results are not significantly different -- the appendix provides the specific numerical results. Note that parameters α and β define the shape of a beta distribution in such a manner that if distribution 1 takes values α=x and β=y, and distribution 2 takes the values α=y and β=x, distribution 2 will be a mirror image of distribution 1 flipped along the line x=.5. Therefore, by analyzing distributions with $c \leq .5$, conclusions can also be drawn for distributions with c on the interval [.5,1].

Method of Moments

Table 1 reports the total absolute deviation for each pair of triangular and beta distribution based on the given mode of the triangular distribution -- provided at the top of each column. A smaller value is preferred when comparing across columns as it indicates less variation between the two CDFs for the corresponding measures used. Table 1 shows that for the analyzed triangular distributions, the mean-variance combination has the lowest AD for each value of c.

Table 1: Absolute Deviation between triangular and beta CDF's for triangular distributions with c ≤ .5

Mode of Triangular (c)	0	0.04	0.08	0.12	0.16	0.2	0.24	0.28	0.32	0.36	0.4	0.44	0.48
Mean-Variance	3.88E-15	0.41	0.72	0.94	1.08	1.15	1.15	1.10	1.00	0.88	0.75	0.66	0.61
Median-Variance	3.88E-15	1.29	1.33	1.34	1.34	1.32	1.26	1.18	1.06	0.92	0.78	0.67	0.61
Mode-Median	0.65	0.93	1.38	1.82	2.23	2.61	2.93	3.21	3.44	3.60	3.70	3.73	3.68
Mode-Variance	3.88E-15	1.47	2.68	3.60	7.65	4.61	4.69	4.49	4.03	3.37	2.53	1.57	0.70
Variance-Skew	3.87E-15	1.42	2.66	3.71	4.53	5.09	5.38	5.35	5.01	4.34	3.36	2.13	0.79
Mean-Median	1.04	0.42	1.03	1.93	2.80	3.64	4.44	5.19	5.90	6.55	7.14	7.67	8.13
Mean-Mode	16.67	1.33	1.58	2.03	2.50	2.94	3.34	3.70	4.01	4.26	4.45	4.59	4.66

To ensure the larger differences were not due to a small number of points with a large deviation, a range of absolute deviation was measured to ensure consistency. If range of absolute deviation is large it could correspond to one or two bad fitting points causing a large absolute deviation despite an otherwise good fit -- this would require further analysis on the outliers. Table 2 shows the range of absolute deviation for each pair of triangular and beta distributions calculated by subtracting the minimum deviation from the maximum deviation. Table 2 shows relative consistency amongst the deviation. Ranges generally fall below .1 with the occasional range of .12 corresponding with larger values in Table 1. The mean-variance combination has all ranges ≤ .03 illustrating the consistency of the best combination based on Table 1. There is one outlier at c=0 for the mean-mode combination -- a value of .24. This value corresponds to the 16.67 absolute deviation at c=0 for the mean-mode combination in Table 1 representing a relatively consistent range of deviation indicative of a poor fitting distribution.

Table 2: Range of Absolute Deviation between triangular and beta CDF's for triangular distributions with $c \leq .5$

Mode of Triangular (c)	0	0.04	0.08	0.12	0.16	0.2	0.24	0.28	0.32	0.36	0.4	0.44	0.48
Mean-Variance	4.44E-16	0.02	0.03	0.03	0.03	0.03	0.02	0.02	0.02	0.02	0.02	0.02	0.01
Median-Variance	4.44E-16	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.02	0.02	0.02
Mode-Median	0.01	0.03	0.05	0.06	0.07	0.08	0.08	0.08	0.08	0.08	0.07	0.07	0.06
Mode-Variance	4.44E-16	0.03	0.06	0.08	0.15	0.09	0.09	0.09	0.08	0.07	0.06	0.04	0.02
Variance-Skew	4.44E-16	0.03	0.06	0.08	0.09	0.10	0.11	0.11	0.10	0.09	0.07	0.05	0.03
Mean-Median	0.022512	0.02	0.04	0.06	0.08	0.09	0.10	0.11	0.11	0.12	0.12	0.12	0.12
Mean-Mode	0.24	0.02	0.03	0.04	0.05	0.06	0.06	0.06	0.06	0.07	0.08	0.08	0.08

Based on results provided in Tables 1 and 2, the combination of mean-variance outperforms the other measure combinations. Equations 3 and 4 provide the general form for calculating α and β using the mean-variance combination.

$$\alpha = \frac{-(\mu * (\mu^2 - \mu + \sigma^2))}{\sigma^2} \tag{3}$$

$$\beta = \frac{\mu - \sigma^2 + \mu * \sigma^2 - 2\mu^2 + \mu^3}{\sigma^2} \tag{4}$$

This combination provides both the lowest absolute deviation as well as the lowest range of absolute deviation -- indicating consistently low results which can be interpreted as a beta distribution most similar to the given triangular distribution. In addition to providing the lowest absolute deviation, there are other advantages of using mean and variance to estimate α and β. The metrics of mean and variance are easily understood by most analysts and core to any form of statistical analysis. While one could argue that the differences between the mean-variance and median-variance are negligible, the exact median of the beta distribution can only be found using the incomplete beta function; otherwise it is only an estimate. Conversely, the exact values of mean and variance are both functions of α and β.

Quantile Estimation

Table 3 compares the results of the decile estimation method to the mean-variance MoM estimate using the total absolute deviation metric previously discussed. The results at various modes of the given triangular distribution shows the decile estimate is only marginally different than the method of moments. For every value of the mode, the greatest difference between using quantile estimation and using method of moments is .08.

Table 3: Comparison of Decile Estimation to Mean-Variance MoM Combination for triangular distributions with $c \leq .5$

Mode of Triangular (c)	0	0.04	0.08	0.12	0.16	0.2	0.24	0.28	0.32	0.36	0.4	0.44	0.48
Mean-Variance	3.88E-15	0.41	0.72	0.94	1.08	1.15	1.15	1.10	1.00	0.88	0.75	0.66	0.61
Range of Mean-Variance AD	4.44E-16	0.02	0.03	0.03	0.03	0.03	0.02	0.02	0.02	0.02	0.02	0.02	0.01
Decile Estimation AD	3.88E-15	0.34	0.64	0.89	1.02	1.09	1.10	1.04	0.95	0.82	0.70	0.61	0.57
Range of Decile Estimation AD	8.32E-31	2.98E-03	0.01	0.01	0.02	0.02	0.11	0.01	0.01	0.01	0.01	0.01	0.00

IV. Empirical Example

While the results section characterizes the differences between individual distributions, cost estimates are often the aggregation of many distributions interacting together. To characterize this aggregate effect of converting to a beta distribution, an empirical example using a Monte-Carlo simulation is conducted. The example uses a scenario where ten different experts are tasked to provide the mode cost of an individual item with a known minimum and maximum -- allowing for the normalized triangular distribution to be used. The experts' input is simulated by drawing ten samples from a continuous uniform distribution over the support [0,1] -- see Table 4. Once the set of ten triangular distributions is defined, a corresponding set of beta distributions is created using the mean-variance method of moments combination. Monte-Carlo simulations consisting of 10,000 iterations are run for both sets of distributions, similar to how a Monte-Carlo simulation would be run to determine a range of possible costs in an estimate. For each iteration for a set of distributions, a number is randomly drawn from each of the 10 distributions and the mean of the 10 draws is then plotted on a histogram. The mean is used rather than the sum to characterize the percentile of the cost since all the distributions are over the support [0,1]. A summation would be a more accurate depiction if the minimum and maximum values varied.

Figures 1 and 2 summarize the results of the simulation using the triangular distribution and the beta distribution, respectively. The simulation using the triangular distributions has a minimum value 2.6 percentage points less than the simulation using the beta distributions as well as a maximum value 1.4 percentage points less than the beta distribution. The modal value on the triangular simulation is less than .03 percentage points greater than the beta distribution's simulation. Furthermore, standard deviations are nearly identical as are the values for the inner 90% of the distribution. The minor differences in the two simulations indicate that there is little difference between using triangular distributions or using method of moments transformed beta distributions.

Table 4: Values of Triangular Distributions used for empirical example

Mode	0.57	0.19	0.82	0.29	0.57	0.45	0.72	0.65	0.4	0.12
Alpha	2.5895	1.6229	2.4503	1.9590	2.5895	2.4038	2.5886	2.6218	2.2842	1.3845
Beta	2.3586	2.4685	1.5887	2.5968	2.3586	2.5696	1.9264	2.1451	2.6105	2.3239

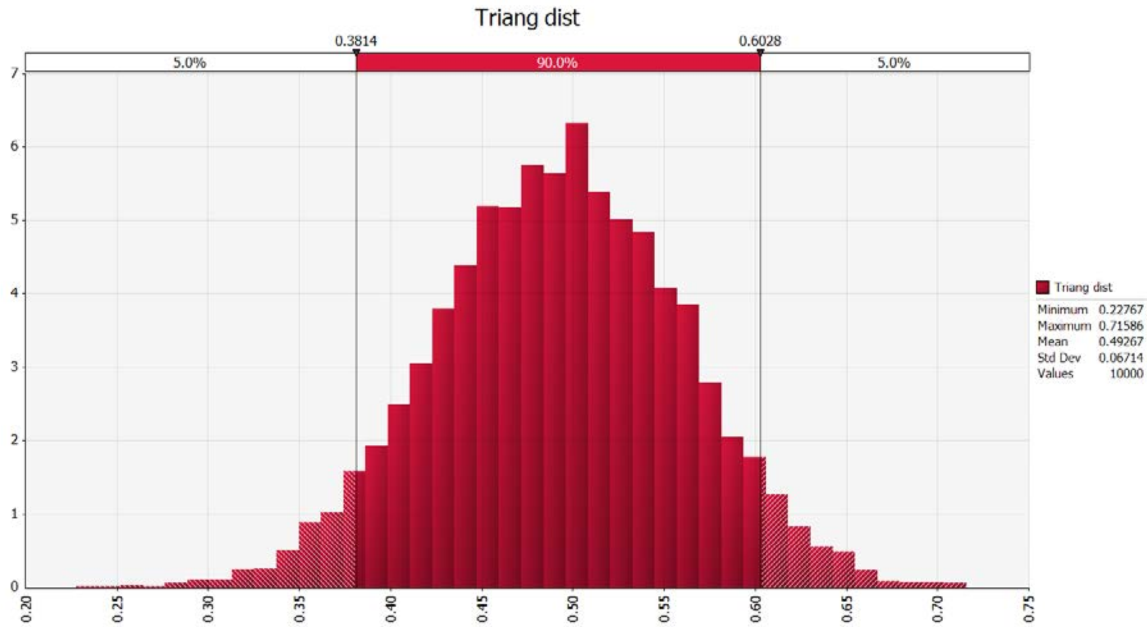


Figure 1: Monte-Carlo Simulation using triangular distribution

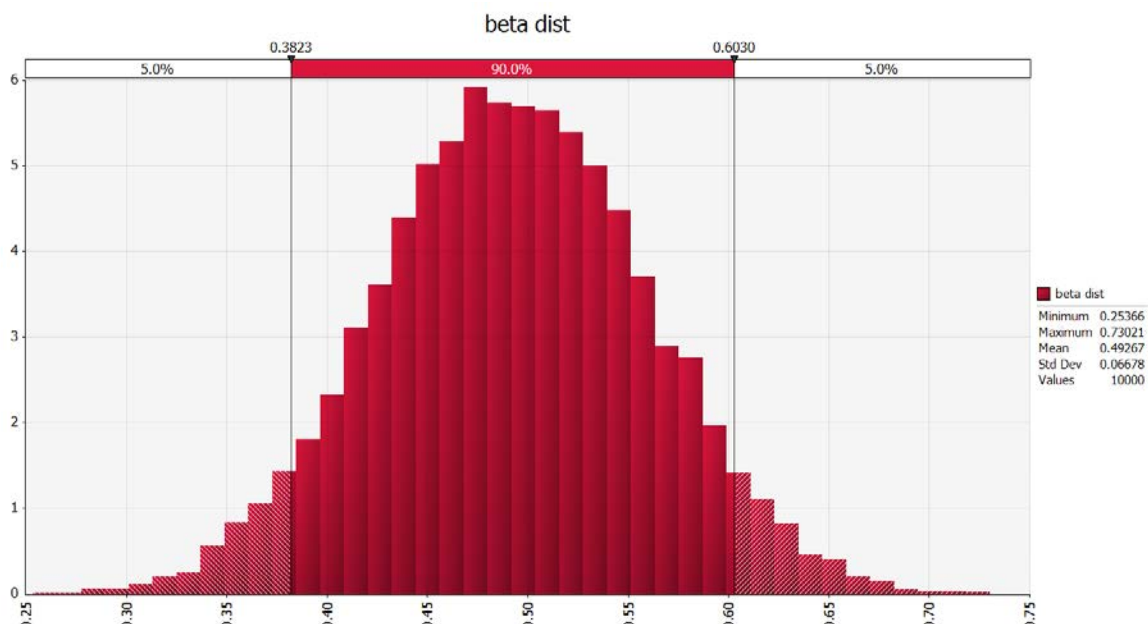


Figure 2: Monte-Carlo Simulation using beta distribution

IV. Conclusion

Results of this study show that using a method of moments using the mean and variance parameters obtained from a given triangular distribution provides a corresponding beta distribution that is similar in fit to the original triangular distribution. When comparing this method to quantile estimation, there is very little difference between the two methodologies. For this reason, it is recommended that analysts use the mean and variance combination of method of moments to estimate parameters α and β in a corresponding beta distribution.

Using this methodology provides several advantages to an analyst. First, the analyst is not required to change their methodology to obtain SME inputs; they can adhere to guidance in JACSRUH and continue eliciting a high, low, and most likely value. Once those values are found and the triangular distribution is defined, creating the beta distribution only requires inputting mean and variance into Equations 3 and 4 to find the parameters which define the beta distribution. Second, this technique does not require advanced knowledge of the beta distribution or statistical knowledge beyond what the analyst should already know through their use of Monte-Carlo simulation with a triangular distribution. This method only requires the analyst to use the equations to estimate parameters and create a new beta distribution to be run in simulations instead of using the triangular distribution.

Given the relative uncertainty surrounding subjective inputs, the differences between the given triangular distribution and the beta distribution with parameters found using equations 3 and 4 are minute enough to suggest the use of the beta distribution for subjective uncertainty. It causes only a small change in the overall cost estimate while allowing the analyst to use the distribution for more advanced techniques such as Bayesian analysis.

Future research may aim to further alleviate the concerns of differences in fit by simplifying more advanced techniques such as quantile estimation and MLE for analysts. While these methodologies may provide only a marginally better fit empirically, if simplified to a level a junior analyst could utilize, a more exact approximation of the beta distribution is preferred. Additionally, empirical studies comparing actual costs to subjective estimates could reveal whether the beta distribution or triangular distribution more accurately predicts costs. While the theoretical difference in distributions is a small percentage, these small percentages amount to great sums when one considers programs cost millions, sometimes billions, of dollars. Finally, empirical studies applying this method to entire cost estimates are recommended as they would help provide insight as to whether this method feasible for predicting costs and maintaining the integrity of an estimate. In summation, while this manuscript provides theoretical evidence to support using a beta distribution rather than a triangular distribution for subjective cost estimation, empirical studies would go a long way to help justify this claim.

Appendix A -- Method of Moments Comparison using MSE

Mode of Triangular (c)	0	0.04	0.08	0.12	0.16	0.2	0.24	0.28	0.32	0.36	0.4	0.44	0.48
Mean-Variance	8.32E-31	0.003	0.009	0.014	0.018	0.019	0.019	0.017	0.015	0.012	0.009	0.007	0.006
Median-Variance	8.32E-31	0.029	0.031	0.031	0.029	0.028	0.025	0.022	0.018	0.015	0.011	0.008	0.006
Mode-Median	5.42E-03	0.012	0.031	0.055	0.083	0.110	0.135	0.155	0.171	0.181	0.185	0.182	0.174
Mode-Variance	8.32E-31	0.029	0.096	0.174	0.829	0.285	0.293	0.269	0.218	0.154	0.089	0.038	0.009
Variance-Skew	8.32E-31	0.027	0.095	0.185	0.276	0.350	0.390	0.388	0.341	0.257	0.157	0.066	0.013
Mean-Median	1.48E-02	0.003	0.020	0.060	0.118	0.189	0.270	0.357	0.447	0.538	0.628	0.714	0.795
Mean-Mode	3.33E+00	0.024	0.034	0.054	0.082	0.113	0.145	0.176	0.204	0.230	0.251	0.268	0.279

Appendix B -- Method of Moments Comparison using Max Deviation

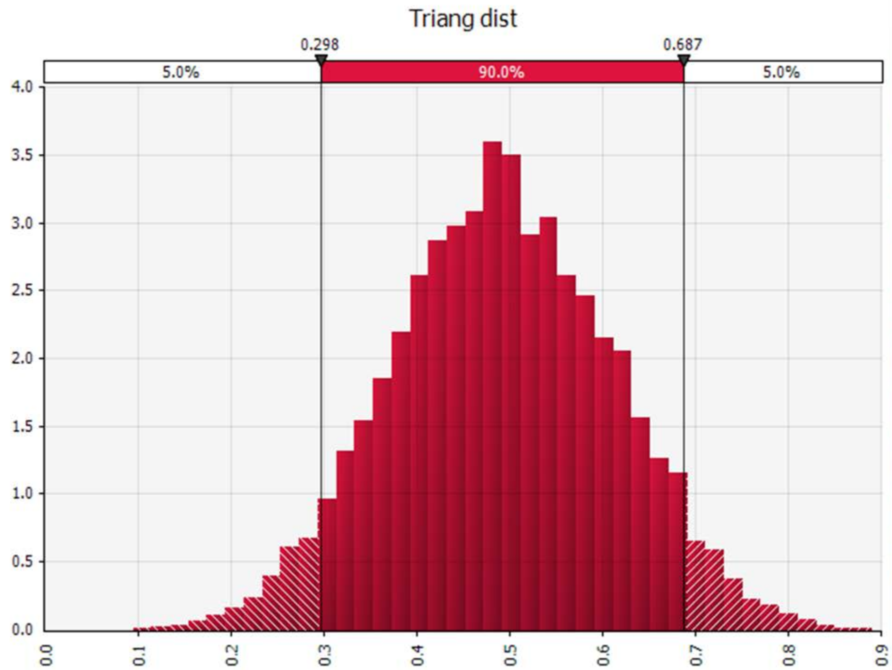
Mode of Triangular (c)	0	0.04	0.08	0.12	0.16	0.2	0.24	0.28	0.32	0.36	0.4	0.44	0.48
Mean-Variance	4.44E-16	0.021	0.028	0.029	0.028	0.025	0.025	0.025	0.024	0.022	0.019	0.017	0.015
Median-Variance	4.44E-16	0.030	0.033	0.033	0.033	0.032	0.032	0.031	0.029	0.027	0.023	0.020	0.016
Mode-Median	1.07E-02	0.028	0.047	0.061	0.070	0.076	0.079	0.081	0.081	0.079	0.075	0.070	0.063
Mode-Variance	4.44E-16	0.033	0.057	0.075	0.152	0.093	0.094	0.089	0.081	0.069	0.055	0.039	0.022
Variance-Skew	4.44E-16	0.032	0.057	0.077	0.092	0.102	0.106	0.106	0.099	0.087	0.070	0.049	0.026
Mean-Median	2.26E-02	0.020	0.043	0.062	0.078	0.091	0.101	0.108	0.114	0.119	0.122	0.124	0.125
Mean-Mode	2.50E-01	0.023	0.028	0.037	0.048	0.056	0.061	0.064	0.065	0.070	0.075	0.080	0.084

Appendix C -- Empirical example with applied correlation

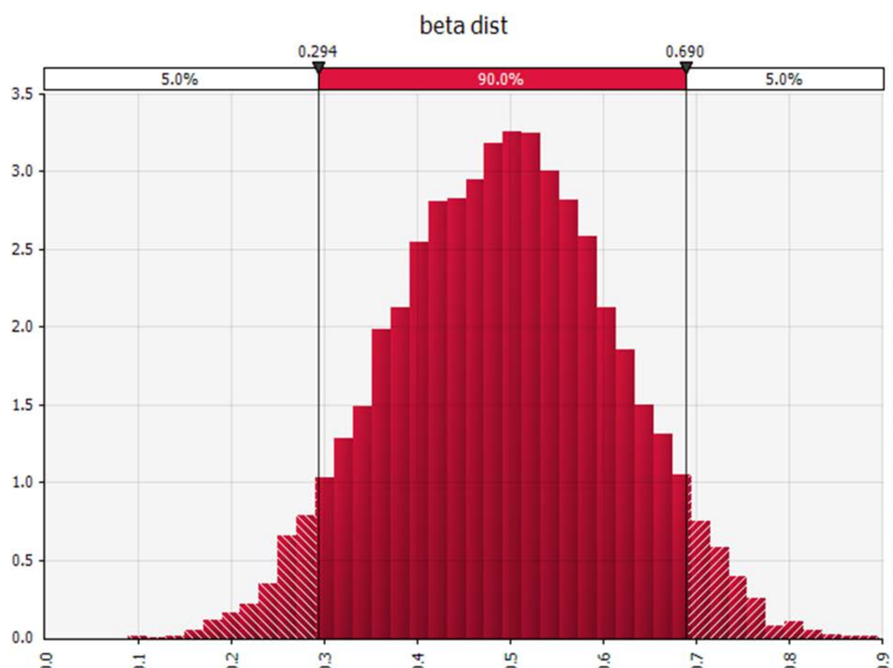
While the empirical example on page 13 provides evidence to suggest there is little difference between in a cost estimate as a whole when considering the aggregate differences in a set of ten triangular distributions and a set of ten beta distributions, it does exclude correlation among the distributions. This is likely not indicative of a cost estimate as a rise in cost of one element (e.g. fuselage) likely corresponds with a rise in cost of another element (e.g. wings). This appendix applies correlation at .25 between the various distributions to compare the effects of the aggregate differences in distributions similar to the example on page 14.

Methodology remains consistent with the example provided in the manuscript to include same modal values for the triangular distributions and same values of α and β for the beta distributions. Once the distributions are defined a correlation matrix is set up to include 25% correlation between each triangular distribution and 25% correlation between each beta distribution. Once the correlation matrix is applied to the distributions, a Monte-Carlo simulation consisting of 10,000 iterations is run for each set of ten distributions (set of triangular and set of beta). For each iteration, one draw is taken from each of the ten distributions and the mean value is plotted on a histogram. Graphical results are shown at the end of this appendix. The minimum/maximum values of the beta distribution simulation are .088/.897 while the minimum for the triangular distribution simulation is .095/.890. Similar to the example without applied correlation, these differences are less than two percentage points (the min/max values actually have less error with applied correlation than without). The mean values are again identical with values of .49267 while the standard deviation values differ by .1 percentage point. Similarly, the inner 90 percentile range is also nearly identical

This simulation suggests there is little difference between the use of the triangular distribution and method of moments estimated beta distribution. Even with correlation applied as would be realistic in a cost model, there is marginal difference between the resulting simulations.



	Triang dist
Cell	Sheet1!F13
Minimum	0.09457
Maximum	0.89039
Mean	0.49267
90% CI	± 0.00195
Mode	0.49294
Median	0.49178
Std Dev	0.11836
Skewness	0.0111
Kurtosis	2.8095
Values	10000
Errors	0
Filtered	0
Left X	0.298
Left P	5.0%
Right X	0.687
Right P	95.0%
Dif. X	0.38939
Dif. P	90.0%
1%	0.22388
5%	0.29759
10%	0.33911
15%	0.36893
20%	0.39182



	beta dist
Cell	Sheet1!I13
Minimum	0.08890
Maximum	0.89677
Mean	0.49267
90% CI	± 0.00196
Mode	0.48691
Median	0.49417
Std Dev	0.11930
Skewness	-0.0098
Kurtosis	2.6716
Values	10000
Errors	0
Filtered	0
Left X	0.294
Left P	5.0%
Right X	0.690
Right P	95.0%
Dif. X	0.39536
Dif. P	90.0%
1%	0.22277
5%	0.29443
10%	0.33505
15%	0.36490
20%	0.38893

Appendix D -- Supplemental Literature Review

Bjerga et al. (2014) conducted work to model uncertainty through their description of the variation in the occurrences in time of a specific event using a Poisson distribution with the purpose of providing clarity in the risk field and risk regulation. The purpose is to help provide a framework to model undesirable events occurring with some sort of frequency in the future. Fallon (1976) wrote a RAND study focusing on cognitive influences of availability, anchoring, and representativeness to determine how they could distort an expert's assessment of uncertainty. The study concluded that to advisers and analysts involved in planning be made aware of the influences which bias their subjective assessments so they can assess their true state of information. Brown (1973) asked 31 students at UCLA to forecast 14 quantities such as GNP, consumer prices, and deaths in South Vietnam and found 95% of respondents gave meaningful distributions that were usable. He further found that the true answer often occurred in the tails of the distributions provided and this issue can be alleviated by combining the individual responses into a consensus distribution. Frick (2010) discusses how the DOD is risk averse with program managers concentrating on foreseeable events with the exclusion of all other possible events. He goes on to discuss how successful business cultivate a culture of risk taking that does not punish honest failure and suggests that the DOD should truly accept and plan for the unknown.

Dorp and Kotz (2002) draw motivation from Johnson (1997) and offer the two-sided power distribution as an alternative to the triangular and beta distributions reasoning that its MLE is computationally straightforward and robust when compared to the MLE of a beta distribution. They additionally argue that is allows for a J-shape and

U-shape making it more diverse than the triangular distribution. Johnson (2002) examined using the Pearson-Tukey mean and standard deviation approximations on unbounded distributions and found the mean approximation is highly accurate while the standard deviation approximation for unbounded distributions is not. He also found that by weighting the 5th, 50th, and 95th percentiles of a triangular distribution approximation one could give a universally accurate mean and standard deviation approximation. Farnum and Stanton (1987) comment that the beta distribution is used because it “provides a rich family of distributional shapes” (Farnum & Stanton, 1987, pp. 287) while examining how a mean is estimated based on a low, high, and most likely values. While their method is useful over a certain range the MoM technique in this paper offers consistency in that it matches the exact mode of the provided triangular distribution provided by the low, high, and most likely parameters.

Parameter estimation has been conducted for a three-parameter beta distribution (two shape parameters and a scaling parameter) for simple unit hydrograph theory. This type of estimation was specific to this field and required peak runoff and time of peak runoff information to estimate the parameters (Bhunya et al., 2004).

References – Paper 1

Berny, J. (1989). A New Distribution Function for Risk Analysis. *The Journal of the Operational Research Society*, *40*(12), 1121-1127. doi: 10.1057/jors.1989.193

Bhunya, K.P., Mishra, K.S., Ojha, S.P.C, & Berndtsson, R. (2004). Parameter Estimation of Beta Distribution for Unit Hydrograph Derivation. *Journal of Hydrologic Engineering*, 9(4), 325-332. doi: 10.1061/(ASCE)1084-0699(2004)9:4(325)

Bjerga, T., Aven, T., & Zio, E. (2014). An illustration of the use of an approach for treating model uncertainties in risk assessment. Reliability Engineering and System Safety, 125, 46–53. doi: 10.1016/j.ress.2014.01.014

Brown, T. (1973). An Experiment in Probabilistic Forecasting. RAND Corporation, R-944-ARPA. https://www.rand.org/content/dam/rand/pubs/reports/2006/R944.pdf

Brownstein, N. C., Louis, T. A., O'Hagan, A., & Pendergast, J. (2019). The Role of Expert Judgment in Statistical Inference and Evidence-Based Decision-Making. *The American statistician*, *73*(1), 56–68. doi: 10.1080/00031305.2018.1529623

Burgman, M. A. (2015). *Trusting Judgements: How to Get the Best Out of Experts, Cambridge*. UK: Cambridge University Press.

Clemen, R. & Reilly, T. (2014). *Making Hard Decisions with Decision Tools (3rd edition)*. South-Western Cengage Learning.

Coleman, R., Braxton, P., & Druker, E. (2010). The Correct Use of Subject Matter Experts in Cost Risk Analysis. *Seventh Annual Acquisition Research Symposium*

Wednesday Session, Volume 1, 312-333. https://www.researchgate.net/profile/Richard_Coleman4/publication/235143544_The_Correct_Use_of_Subject_Matter_Experts_in_Cost_Risk_Analysis/links/55c8bb7108aeb97567476a95/The-Correct-Use-of-Subject-Matter-Experts-in-Cost-Risk-Analysis.pdf

Davino, C., Marilena, F., & Vistocoo, D. (2014). *Quantile Regression Theory and Application.* John Wiley & Sons Ltd.

Dorp, J. & Kotz, S. (2002). A Novel Extension of the Triangular Distribution and Its Parameter Estimation. *Journal of the Royal Statistical Society. Series D (The Statistician),* 51(1), 63-79. doi: 10.1111/1467-9884.00299

Fallon, R. (2006). Subjective Assessment of Uncertainty. RAND Corporation, P-5581. https://www.rand.org/pubs/papers/P5581.html.

Farnum, N. & Stanton, L. (1987). Some Results Concerning the Estimation of Beta Distribution Parameters in PERT. *Journal of Operational Research Society,* 38(3), 287-290. doi: 10.1057/jors.1987.45

Frick, D. (2010). Embracing Uncertainty in DoD Acquisition. *Defense Acquisition Universtiy.* https://apps.dtic.mil/dtic/tr/fulltext/u2/a523878.pdf

Helton, J.C. (1997). Uncertainty and Sensitivity Analysis in the Presence of Stochastic and Subjective Uncertainty, *Journal of Statistical Computation and Simulation,* 57(1-4), 3-76. doi: 10.1155/2012/617481

Johnson, D. (1997). The Triangular Distribution as a Proxy for the Beta Distribution in Risk Analysis. *Journal of the Royal Statistical Society. Series D (The Statistician),* 46(3), 387-398. doi: 10.1111/1467-9884.00091

Johnson, D. (2002). Triangular Approximations for Continuous Random Variables in Risk Analysis. *The Journal of the Operational Research Society,* 53(4), 457-467. doi: 10.1057/palgrave.jors.2601330

Jorgensen, M. (2004). A Review of Studies on Expert Estimation of Software Development Effort, *The Journal of Systems and Software*, 70(1-2), 37-60. doi:10.1016/S0164-1212(02)00156-5

Kerman, J. (2011). *A Closed-Form approximation for the median of the beta distribution* [Working paper]. https://arxiv.org/pdf/1111.0433.pdf

Machina, M., & Schmeidler, D. (1992). A More Robust Definition of Subjective Probability. *Econometrica*, *60*(4), 745-780. doi:10.2307/2951565

O'Hagan, A. (2019). Expert Knowledge Elicitation: Subjective but Scientific. *The American Statistician*. 73(sup1), 69-81. doi: 10.1080/00031305.2018.1518265

Smith, A., McDowell, J., & Fichter, L. (2010). U.S. Air Force Cost Risk and Uncertainty Analysis Metrics Manual (CRUAMM). https://www.ncca.navy.mil/tools/csruh/CRUAMM%20Version%2016Nov2011%20with%20Preface%2005April2013.pdf

Thomas, D. & Fitch, J. (2014). Joint Agency Cost Schedule Risk and Uncertainty Handbook. https://www.ncca.navy.mil/tools/csruh/JA_CSRUH_16Sep2014.pdf

Wang, J. (2005). A note on the estimation in the four-parameter beta distribution. *Communication in Statistics-Simulation and Computation*, 34(3), 495-501. doi: 10.1081/SAC-200068514

Whyte, G., & Classen, S. (2012). Using storytelling to elicit tacit knowledge from SMEs. *Journal of Knowledge Management*, *16*(6), 950-962. doi: http://dx.doi.org/10.1108/13673271211276218

Williams, T. (1992). Practical Use of Distributions in Network Analysis. *The Journal of the Operational Research Society*, 43(3), 265-270. doi:10.2307/2583716

I. Introduction

Bayesian updating is a type of statistical analysis that allows one to use prior information in combination with a new piece of data in order to make inferences -- or develop an updated distribution around some uncertain parameter. Bayesian updating has been applied to a variety of fields to include engineering (Lyngdoh et al., 2019), Seismic Hazard Assessments (Vialett et at., 2019), and research on price elasticity of demand (Sun et al., 2016).

In the field of finance Hwang finds a Bayesian probability approach more effective for predicting the errors of cost estimates (Hwang, 2013). Specific implementations have been applied to risks on completion costs of construction projects (Namazian & Yakhchali 2018) and software engineering cost models where it was found Bayesian updated subjective uncertainty elicited via delphi technique combined with sample data better estimates software costs (Chulani et al., 1999). Additionally, through their use of a log-normal distribution as a conjugate prior Caron et al. conclude “The use of a Bayesian approach, based on expert opinion elicitation, permits the exploitation of subjective judgments in a rigorous and formal way leading to an improvement in accuracy of estimates at completion within an [earned value management] framework” (Caron et al., 2013, pp. 15). This manuscript seeks to expand on the work of Bayesian updating in the field of cost estimation by applying a framework for using a beta distribution as a conjugate prior to update subjective uncertainty.

Specifically, this manuscript considers two instances in which Bayesian updating can be applied in the field of cost estimation. First, applying it at a macro level estimate to characterize the effects on budgeting in order to account for underestimation in the DoD. When a cost estimate is developed it provides a range of possible costs using a distribution. Budgets are set at some percentile of said distribution. As a program continues additional information is obtained and able to be used as an input -- usually in the form of sample data such as cost of the first lot of production. This information can be used along with the prior information (i.e. the prior distribution) to create a Bayesian updated cost estimate and provide decision makers a tool for budgeting. The characterization can help alleviate the issue of cost overruns due to errors in cost estimates.

The second application of Bayesian updating is combining multiple expert opinions into a single distribution. Expert opinion is often used in the field of subjective uncertainty with various studies exploring best practices (O'Hagan, 2019, Burgman, 2015, Grafton & Selwyn, 2012). Specific implementation has been done in the software field to provide guidelines for estimation using expert opinion (Jorgensen, 2004). These elicitations are turned into distributions indicative of potential costs. However, there are times where either multiple experts, or a single expert at different times, provide multiple distributions. Coleman et al. (2010) provides a method for combining these multiple distributions in a single distribution. Bayesian updating provides an alternative method to Coleman et al. for combining multiple distributions.

II. Methodology

Effects on Budget

RAND Corporation found a mean cost growth (defined as the ratio between the most recent selected acquisition report (SAR) and the cost estimate baseline reported in a prior SAR issued at the time of a given milestone) of 46% in the DoD (Arena et al., 2006). When analyzing sources of cost growth, 10.1% is attributed to errors in cost estimates (Arena et al., 2008). Uncertainty analysis in cost estimates impacts budget requests -- and thus cost overruns. Giving cost estimators tools to use new information to update budget requests could prove valuable for limiting cost growth due to errors in estimates. By comparing the differences of budgeting under various levels of uncertainty before and after applying Bayesian updating, changes in budget requests can be compared; offering a technique to account for errors in cost estimates.

Uncertainty falls into two categories, subjective and objective. While objective uncertainty is based around empirical data, subjective uncertainty relies on expert elicitation. The Joint Agency Cost Schedule Risk and Uncertainty Handbook (JACSRUH) recommends using a triangular distribution for subjective uncertainty (Thomas & Fitch, 2014). This analysis considers 11 triangular distributions, fixed on the interval [0,1], with various modes to represent subjective uncertainty elicited from SMEs. The 11 triangular distributions are then converted to corresponding beta distributions with a method-of-moments techniques using the mean and variance population parameters of the triangular distribution. This transformation to a beta distribution enables the use of Bayesian updating. Fixing the triangular distribution to the interval

[0,1] allows for a comparison that is based on percent change in budget and allows the use of the two-parameter beta distribution rather than the four-parameter.

Bayesian updating is a statistical method applying Bayes theorem to update a probability distribution as more information becomes available. Bayes theorem states

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)} \tag{1}$$

where P(B|A) is the likelihood and P(A) is the prior probability – referred to as a conjugate prior. Since P(B) is a scaling factor Bayes theorem can be proportionally written as

$$P(A|B) \propto P(B|A) * P(A) \tag{2}$$

In the specific case being used for this comparison the beta distribution is the conjugate prior -- the distribution used prior to Bayesian updating. The Bernoulli distribution is the likelihood function used to update the beta distribution. A binomial distribution follows the posterior parameters prediction

$$\alpha_{t+1} = \alpha_t + \sum_{i=1}^{n} x_i \qquad \beta_{t+1} = \beta_t + n - \sum_{i=1}^{n} x_i \tag{3}$$

where α_t and β_t are the shape parameters defining the conjugate prior beta distribution, α_{t+1} and β_{t+1} are the shape parameters defining the posterior beta distribution, and x_i is the random variable generated from a Bernoulli trial as being a success or failure. Since the Bernoulli distribution is a single is just a single trial of the binomial distribution (i.e. n=1), in the case of a success (x_i=1) equation 3 reduces to:

$$\alpha_{t+1} = \alpha_t + 1 \qquad \beta_{t+1} = \beta \tag{4}$$

And in the case of failure (x_i=0):

$$\alpha_{t+1} = \alpha_t \qquad \beta_{t+1} = \beta_t + 1 \tag{5}$$

Or simply stated, in the case of success the shape parameter α will increase by a value of 1 and in the case of a failure, the parameter β will increase by a value of 1.

Once parameters for both the conjugate prior beta distribution and Bayesian updated beta distribution are defined, the distributions can be compared at various percentiles to characterize the effect on budget planning. This analysis considers a data point greater than the mode of the beta distribution to equate to a Bernoulli trial where $x_i=1$. However, decision makers can determine their own criteria for the Bernoulli trial and future studies to determine the probability value of p in the Bernoulli trial are recommended. The characterization of the posterior distribution is only based on whether the Bernoulli trial is considered a success or a failure and not the criteria determining success or failure.

After examining the change in budget due to Bayesian updating, a weighted average of expected change in budget is taken to reflect the percent of time $x_i=1$. This study assumes the percentage of time when $x_i=1$ (i.e. the data point received is greater than the mode) is strictly greater than 50% based on a Defense Industry study finding majority of Major Defense Acquisition Programs (MDAPs) experience cost-overruns (Hofbauer et al., 2011) Further studies found one in eight estimates are too low (Coleman et al., 2009). The different proportions of $x_i=1$ and $x_i=0$ are then systematically varied to determine expected budgets. The expected budget change corresponds to the increase in budget for the cost components that are based on subjective uncertainty. This increase in budget could help alleviate some of the errors in cost estimates leading to cost growth. Future studies to empirically calculate a weighted average of budget changes using actual estimates and incoming data are recommended to determine the exact percent of time $x_i=1$.

Combining Multiple Inputs

Coleman, Braxton and Druker (2010) conducted a study on combining multiple subject matter experts' inputs into a single distribution by averaging parameters from triangular distributions. They found that this method is usable when the underlying probability distribution is assumed to be unimodal and there exist subject matter experts who can define the underlying distribution with some degree of accuracy. This scenario is realistic in the world of subjective cost estimating as each element is often estimated through elicitation of a subject matter expert and transformed into a triangular distribution. However, this method has a weakness in that it assumes each expert opinion to be of equal weight. In cases where one expert provides their input at a later time period and possesses additional knowledge equal weighting would not be appropriate. While it is possible to weight the combination of parameters, this adds an additional piece of subjective uncertainty in determining the proper weighting for each expert. This section aims to alleviate this additional uncertainty by providing an alternative way of combining expert inputs through use of Bayesian updating and comparing it to using the Coleman et al. method.

There are three assumptions made for the use of Bayesian updating to be applicable when applying it to combining expert opinions. First, two experts provide their input in the form of two separate distributions -- treated as given. Second, expert two provides their distribution after expert one. Third, an adequate amount of time has elapsed between the times the distributions are obtained such that expert two's distribution is deemed to have some form of additional knowledge.

Similar to the Coleman et al. approach, this study uses sets of two different triangular distributions. Assuming there is a single cost being estimated, expert one offers their input on a low, high, and most likely cost; fully defining a triangular distribution. Expert two offers the same input at a later date and is assumed to have additional knowledge informing their distribution. The Coleman et al. method simply requires averaging the highs, lows, and modes in order to create a single distribution. However, knowing the second input came at a later date and wanting to avoid the subjectivity of weighting the Coleman et al. method, the experts' distributions are converted to beta distribution through a method-of-moments technique. Bayesian updating is then applied to expert one's distribution using the Beta-Bernoulli methodology discussed in the previous section of this manuscript where a success is defined as the mode of the second beta distribution being greater than the mode of expert one's beta distribution. Mode is chosen because it is a metric directly provided by the expert during elicitation. While other measures such as mean or median could be derived, mode is used because it is directly provided by the expert. Since we are using the original distribution as the conjugate prior its shape parameters α and β equate to α_t and β_t. The min and max of the distribution is found by averaging from the original two distributions similar to Coleman et al. in order to derive the four parameters necessary to define a four-parameter beta distribution. This technique is used across multiple combinations of triangular distributions before comparing the resulting Bayesian beta distribution to that of the triangular output using the Coleman et al. approach. The distributions are then characterized in terms of stochastic dominance (i.e. the percent of time the CDF of one distribution is greater than the CDF of another) to analyze which would predict higher costs. Future research on this application would involve an empirical study to determine which method of combining multiple distributions best fits actual costs being predicted.

III. Results

Effects on Budget

Eleven distributions are analyzed fixed on the domain [0,1] with the mode varying along the same domain at .1 intervals. The 11 distributions are considered to have an adequately diverse range of skewness to be representative of potential cost estimates and allow the results to adequately characterize the effect on budgeting. To adequately capture the types of risk tolerances decision makers may have when budgeting under uncertainty, CDF values at the 30th, 50th, and 70th percentiles are compared for each of the 11 distributions. If a decision maker is budgeting at the 30th percentile they are characterized as risk takers, while the 50th percentile is characterized as risk neutral, and the 70th percentile is characterized as risk averse. Tables 1 and 2 show the effects of Bayesian updating on the various distributions at the different percentiles - Table 1 assumes a success (x_i=1) for the Bernoulli trial while Table 2 assumes a failure (x_i=0).

Table 1: Differences in CDF's for beta distributions when Bernoulli trial x_i=1

Risk Aversion Level	c=0	Updated	c=.1	Updated	c=.2	Updated	c=.3	Updated	c=.4	Updated	c=.5	Updated
0.3	0.163	0.363	0.212	0.378	0.258	0.399	0.301	0.424	0.341	0.452	0.379	0.482
0.5	0.293	0.500	0.340	0.505	0.383	0.519	0.423	0.539	0.462	0.564	0.500	0.593
0.7	0.452	0.637	0.486	0.632	0.519	0.638	0.552	0.652	0.586	0.672	0.621	0.698
Risk Aversion Level	**c=.6**	**Updated**	**c=.7**	**Updated**	**c=.8**	**Updated**	**c=.9**	**Updated**	**c=1**	**Updated**	**Average**	
0.3	0.414	0.514	0.448	0.549	0.481	0.585	0.514	0.625	0.548	0.669	**0.125**	
0.5	0.538	0.626	0.577	0.662	0.617	0.702	0.660	0.746	0.707	0.794	**0.114**	
0.7	0.659	0.729	0.699	0.764	0.742	0.803	0.788	0.845	0.837	0.888	**0.093**	

Table 2: Differences in CDF's for beta distribution when Bernoulli trial x_i=0

Risk Aversion Level	c=0	Updated	c=.1	Updated	c=.2	Updated	c=.3	Updated	c=.4	Updated	c=.5	Updated
0.3	0.163	0.112	0.212	0.155	0.258	0.197	0.301	0.236	0.341	0.271	0.379	0.302
0.5	0.293	0.206	0.340	0.254	0.383	0.298	0.423	0.338	0.462	0.374	0.500	0.407
0.7	0.452	0.331	0.486	0.375	0.519	0.415	0.552	0.451	0.586	0.486	0.621	0.518
Risk Aversion Level	**c=.6**	**Updated**	**c=.7**	**Updated**	**c=.8**	**Updated**	**c=.9**	**Updated**	**c=1**	**Updated**	**Average**	
0.3	0.414	0.328	0.448	0.348	0.481	0.362	0.514	0.368	0.548	0.363	**-0.093**	
0.5	0.538	0.436	0.577	0.461	0.617	0.481	0.660	0.495	0.707	0.500	**-0.114**	
0.7	0.659	0.548	0.699	0.576	0.742	0.601	0.788	0.622	0.837	0.637	**-0.125**	

As a practical example of what x_i=1 depicts, consider a decision maker defining the success of the Bernoulli trial as receiving a data point greater than the mode of the prior beta distribution. Table 1 assumes the data point is larger than the mode and shows the effect on budgeting using the posterior distribution. To interpret the results, consider the risk aversion level 0.3 and the first two green columns representing a modal value of 0 and its corresponding posterior distribution when x=1 for the Bernoulli trial. Using the original distribution, a decision maker who is a risk taker would budget at the 16.3% level of the range of possible costs to ensure there is only a 70% chance of cost overruns. Using the posterior distribution, the decision maker would budget at the 36.3% level to ensure the same risk of cost overrun. This means that a Bernoulli trial of x_i=1 would result in a budgetary increase of 20%. Table 1 displays the same scenarios for risk neutral and risk averse decision makers across various modes on a .1 interval. On average, the risk taker would increase their budget 12.5%, the risk neutral decision maker would increase their budget 11.4%, and the risk averse decision maker would increase their budget 9.3%.

Table 2 shows similar results to Table 1 but instead operates under the assumption that x_i=0. That is, the data point received was below the mode. Results are the inverse of Table 1 in that the average columns show budgetary decreases for the risk averse decision maker equal to that of the budgetary increase of the risk taker. The risk neutral decision maker changes to a budgetary decrease of the same magnitude as Table 1 in the case of x_i=0.

The expected budget change is calculated in Table 3 by taking a weighted average to reflect the percent of time x_i=1. Budgeting at the median, the expected budget increase when x_i=1 is 4.55%. Applying this method at the extremes, a risk-averse decision maker (risk aversion level=.7) would have an expected budget increase of .54% if x_i=1 in 60% of instances. While a risk-taking decision maker (risk aversion level=.3) has an expected budget increase of 8.18%. However, given the results of Coleman et al. (2009) the 80% column may be more indicative of the correct weighting. In this scenario, budgetary increases for subjective range from 4.89% to 8.18%. When discussing DoD programs acquired for millions of dollars, these increases in budget could mitigate some of the error in cost estimates and allow for better planning by decision makers.

Table 3: Expected Budgetary increase based on various percentages of x_i=1

Risk Aversion Level	60%	65%	70%	75%	80%
0.3	3.82%	4.91%	6.00%	7.09%	8.18%
0.5	2.27%	3.41%	4.55%	5.68%	6.82%
0.7	0.54%	1.63%	2.71%	3.80%	4.89%

Combining multiple inputs

Similar to the Coleman et al. (2010) method this study assumes two distributions provided by two subject matter experts. The distributions provided by the two experts do not overlap in order to resemble the distributions of Coleman et al. It also provides visual clarity when analyzing the results graphically. However, overlapping the two distributions would change only the minimum and maximum values and not the shape of the distribution meaning the results would remain consistent with this manuscript's results. The first distribution is fixed on the interval [200,400] with c varying at values of 250, 300, 350. The second distribution is fixed on the interval [500,700] with c varying at values of 550, 600, 650. The c values are chosen to represent positive, neutral, and negative skewness. The two distributions are then systematically varied to represent the inputs of expert one and expert two.

Consider the symmetrical case (Figure 1). The green distribution represents expert one's input while the red represents expert two's input. The dark red distribution uses Coleman et al. to combine the distributions resulting in a symmetrical triangular distribution with a minimum of 350 and a maximum of 550. This zero-skewness distribution does not accurately reflect the additional information expert two may have when providing their input. Conversely, the blue distribution is the resulting posterior beta distribution resulting from Bayesian updating. This distribution better represents information provided by expert two while still considering the fact that expert one's information is relevant. The Bayesian updated distribution is skewed left and is stochastically dominated by the Coleman et al. method through the 97th percentile -- predicting a higher chance for higher costs. This prediction for higher costs better reflects the additional information provided in expert two's opinion. Figure 2 provides the reverse scenario where expert one (green) provided a distribution with high costs followed by expert two providing a distribution representing low costs (red). Additionally, in either case the mode is shifted by 17 in the direction of expert two's input.

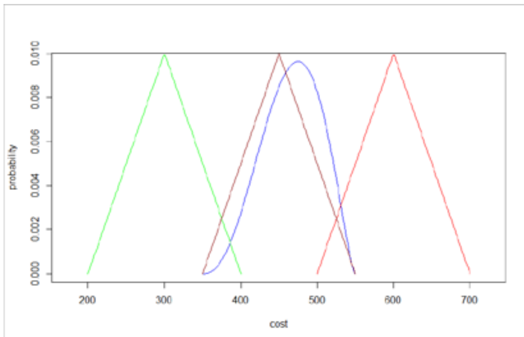


Figure 1: green-expert 1's input, red-expert 2's input, dark red-average triangular parameters, blue-Bayesian updated combination

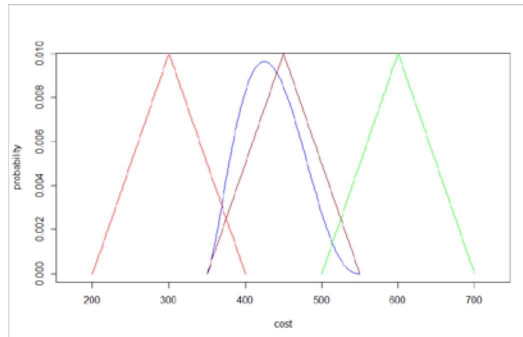


Figure 2: green-expert 1's input, red-expert 2's input, dark red-average triangular parameters, blue-Bayesian updated combination

The remaining eight scenarios are summarized in Table 4 to include the percentage of the domain each distribution stochastically dominates.

Table 4: Stochastic dominance comparison for Coleman et al. method and Bayesian updating method

	Triangular Stochasic Dominance	Beta Stochastic Dominance
Figure 5	85%	15%
Figure 6	100%	0%
Figure 7	100%	0%
Figure 8	94%	6%
Figure 9	0%	100%
Figure 10	0%	100%
Figure 11	6%	94%
Figure 12	15%	85%

Figures 3-6 represent cases where the triangular distribution from the Coleman et al. method stochastically dominates the posterior beta distribution. Common in these figures is expert two providing a distribution with values greater than that of expert one. The beta distribution being stochastically dominated by the triangular results in a prediction of higher costs from the beta distribution. The magnitude of the shift of the mode corresponds to the skewness of the original triangular distributions. In Figure 3 expert two provides a distribution with the opposite skewness as expert one leading to a symmetrical triangular distribution using the Coleman et al. method that aligns more with the corresponding beta distribution. This is because in both cases expert 2 provided a higher distribution for cost shifting the distribution more towards a neutral skewness -- with the Bayesian updated model putting more weight on expert 2's input. However, in Figure 4 expert two provides a distribution with the opposite skewness of expert one resulting in a symmetrical triangular distribution for the Coleman et al. method. Since the Bayesian process is not affected by skewness of distribution -- only the data generating function -- the resulting beta distribution is skewed left to reflect that expert two provided a higher input and properly weight the relevance of their input. The same effect happens in Figures 5-6 where the skewness of the experts' distributions effects the difference in the mode. These predictions align better with the narrative that expert two's additional information provides a higher cost while still considering the fact that expert one's input is relevant.

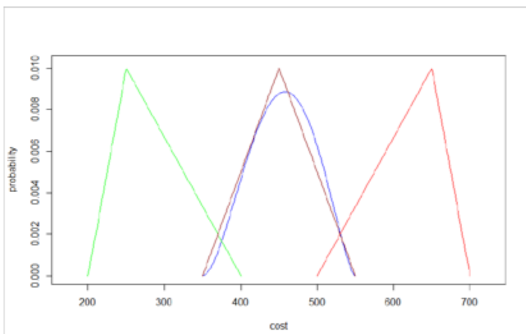


Figure 3: green-expert 1's input, red-expert 2's input, dark red-average triangular parameters, blue-Bayesian updated combination

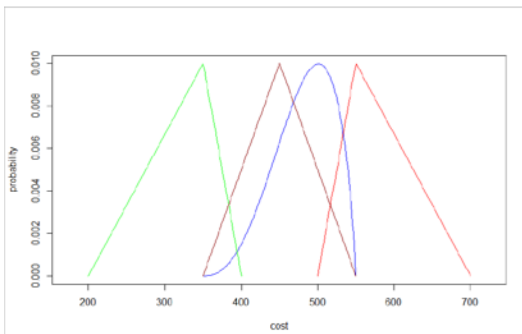


Figure 4: green-expert 1's input, red-expert 2's input, dark red-average triangular parameters, blue-Bayesian updated combination

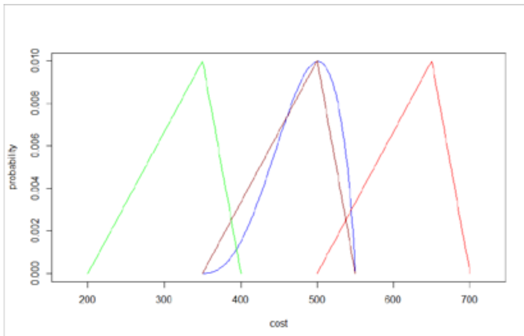


Figure 5: green-expert 1's input, red-expert 2's input, dark red-average triangular parameters, blue-Bayesian updated combination

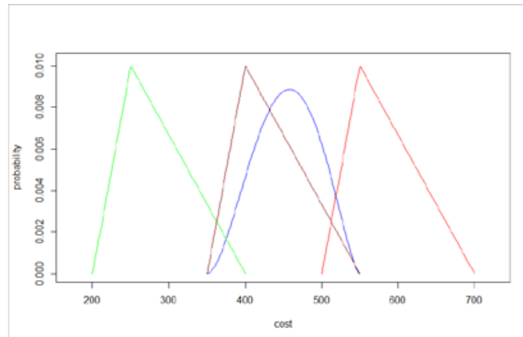


Figure 6: green-expert 1's input, red-expert 2's input, dark red-average triangular parameters, blue-Bayesian updated combination

Figures 7-10 represent cases where the triangular distribution from the Coleman et al. method is stochastically dominated by the posterior beta distribution. Common in these figures is expert two providing a distribution with values less than that of expert one. The beta distribution stochastically dominating the triangular results in a prediction of lower costs from the beta distribution. Again, the effect of the skewness of the expert's distributions is evident in the resulting modal differences. This prediction aligns better with the narrative as expert two's additional information provides a lower cost while still considering the fact that expert one's input is relevant.

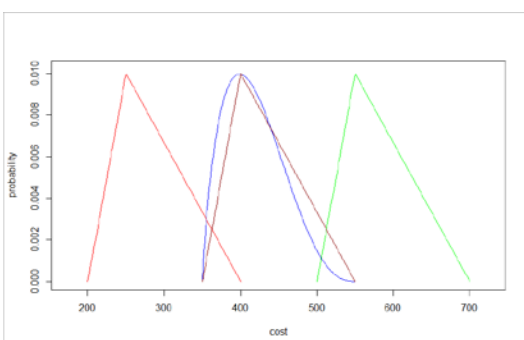


Figure 7: green-expert 1's input, red-expert 2's input, dark red-average triangular parameters, blue-Bayesian updated combination

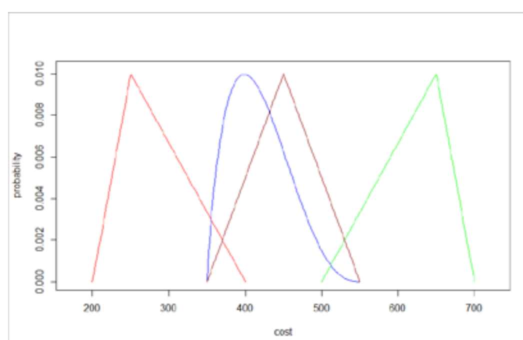


Figure 8: green-expert 1's input, red-expert 2's input, dark red-average triangular parameters, blue-Bayesian updated combination

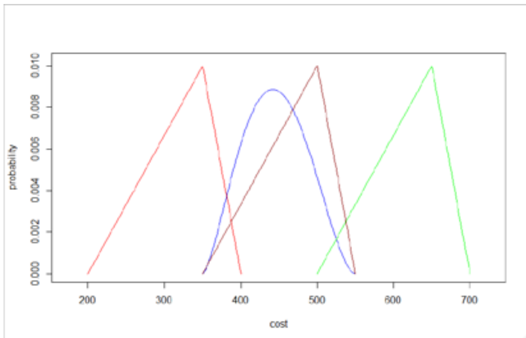


Figure 9: green-expert 1's input, red-expert 2's input, dark red-average triangular parameters, blue-Bayesian updated combination

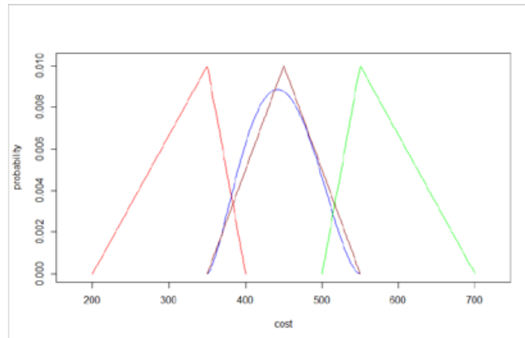


Figure 10: green-expert 1's input, red-expert 2's input, dark red-average triangular parameters, blue-Bayesian updated combination

The Bayesian updating methodology provides decision makers a way to weight the inputs of expert opinions in a less subjective method than weighting individual parameters, while also still considering that the original expert may be correct. Additionally, it is less effected by the skewness of the new information provided but rather uses information resulting from the data-generating function which results in a combined distribution with a mode more indicative of the scenario.

IV. Conclusion

Applying Bayesian updating to uncertainty can have a major impact in the field of cost estimation. This manuscript highlights two potential uses for Bayesian updating – macro level budgeting and combining expert opinion. Analysis on the application of Bayesian updating for macro level budgeting provides decision makers a potential tool to account for underestimates. The use of Bayesian updating to combine expert opinions into a single distribution illustrates an alternative method to Coleman et. al.; a method which more accurately fit scenarios where expert two has some form of additional knowledge and therefore should incorporate the time element Bayesian updating uses.

Both of these methods rely on the subjectivity that is defining the success of a Bernoulli trial being accurate. However, these methods remove subjectivity in redefining distributions to base budgets requests off and weighting individual experts' opinions.

Overall, Bayesian updating for uncertainty analysis can be a powerful tool that allows for the inclusion of additional information such as time at which data is received. If decision makers apply Bayesian updating in the proper context, it could potentially lead to altering decisions which may better fit the situation being analyzed. Further analysis on this subject would involve applying the methods in this manuscript to historical estimates which have actual costs available and comparing the estimate to actuals to see if the Bayesian framework better fits the results.

Appendix A -- Supplemental Literature Review

Duran and Booker (1988) examine sensitivity analysis using the beta distribution as a conjugate prior and provide support for a beta-binomial (and thus Bernoulli) combination stating that it is “mathematically tractable”.

Hammitt and Zhang (2012) examined five methods for combining expert opinion: equal-weight, best-expert, performance, frequentist, and copula. They find with the exception of equal weighting, all the methods require information on the quality of the experts which can be evaluated through seed variables so long as these seed variables are predictive of the expert’s performance on what they are estimating. This supports the use of Bayesian updating as does not require defining seed variables which may or may not be accurate. Cook (1991) also notes the difficulty and different weighting methodologies of combining multiple subject matter expert opinions

Generally, research regarding Bayesian analysis has focused on specific applications, most notably in the field of reliability rather than subjective uncertainty. Weber et al. (2012) found that research works and applications for Bayesian networks in risk analysis, dependability, and maintenance have shown a significant upward trend from 2000-2008 to the degree of an 800% increase in publications. The paper does conclude that a weak point is there is no specific guidance to ensure the model’s coherence. Pollino et al. (2007) acknowledge there is little formal guidance on how to combine data and elicitation in a Bayesian network but seek to provide a detailed methodology through a case study.

Su et al. (2012) applied Bayesian Network to reliability analysis for the first time reasoning that current methods could not accurately describe the status of time-related events. They conclude a dynamic Bayesian network inherits the advantages of static networks while being able to accurately depict an evolutionary process.

Marquez et al. (2010) used Bayesian networks to perform reliability analysis of complex systems in a unifed way while Langseth and Portinale (2007) discussed general properties of Bayesian networks that make them well suited for reliability applications.

References – Paper 2

Arena, M., Leonard, R., Murray, S., & Younossi, O. (2006). Historical Cost Growth of Completed Weapon System Programs. RAND Corporation, TR-343-AF. https://www.rand.org/pubs/technical_reports/TR343.html

Arena, M., Bolten, J., Leonard, S., Sollinger, J., & Younossi, O. (2008). Sources of Weapon System Cost Growth. RAND Corporation, MG-670-AF. https://www.rand.org/content/dam/rand/pubs/monographs/2008/RAND_MG670.pdf

Burgman, M. A. (2015). *Trusting Judgements: How to Get the Best Out of Experts, Cambridge*. UK: Cambridge University Press.

Caron, F., Ruggeri, F., & Merli, A. (2013). A Bayesian Approach to Improve Estimate at Completion in Earned Value Management. *International Journal of Project Management,* 34(8), 3-16. doi: 10.1016/j.ijproman.2016.09.007

Coleman, R., Braxton, P., & Druker, E. (2010). The Correct Use of Subject Matter Experts in Cost Risk Analysis. *Seventh Annual Acquisition Research Symposium Wednesday Session, Volume 1,* 312-333. https://www.researchgate.net/profile/Richard_Coleman4/publication/235143544_The_Correct_Use_of_Subject_Matter_Experts_in_Cost_Risk_Analysis/links/55c8bb7108aeb97567476a95/The-Correct-Use-of-Subject-Matter-Experts-in-Cost-Risk-Analysis.pdf

Coleman, R., Druker, E., Braxton, P., Cullis, B., & Kanick, C. (2009). What Percentile Are We at Now (and Where Are We Going?). *ISPA/SCEA Joint Annual*

Conference and Training Workshop. http://www.iceaaonline.com/ready/wp-content/uploads/2017/09/RS004-Coleman.pdf

Chulani, S., Boehm, B., & Steece, B. (1999). Bayesian Analysis of Empirical Software Engineering Cost Models. *IEEE Transactions on Software Engineering*, 22(4), 573-583. doi: 10.1109/32.799958

Cook, R. (1991). *Experts in Uncertainty*. Oxford University Press.

Duran, S.B. & Booker, M.J. (1988). A Bayes sensitivity analysis when using the beta distribution as a prior. *IEEE Transactions on Reliability*, 37(2), 239-247. doi: 10.1109/24.3748

Hammitt, J. K., & Zhang, Y. (2013). Combining Experts' Judgments: Comparison of Algorithmic Methods Using Synthetic Data. Risk Analysis: An International Journal, 33(1), 109–120. doi: 10.1111/j.1539-6924.2012.01833.x

Hofbauer, J., Sanders, G., Ellman, J., & David, M. (2011). Cost and Time Overruns for Major Defense Acquisition Programs. *Center for Strategic and International Studies. DTIC ADA530059*. https://csis-prod.s3.amazonaws.com/s3fs-public/legacy_files/files/publication/110517_DIIG_MDAP_overruns.pdf

Hwang, S. (2016). A Bayesian Approach for Forecasting Errors of Budget Cost Estimates. *Journal of Civil Engineering and Management*, 22(2), 178-186. doi: 10.3846/13923730.2014.897981

Jorgensen, M. (2004). A Review of Studies on Expert Estimation of Software Development Effort, *The Journal of Systems and Software*, 70(1-2), 37-60. doi:10.1016/S0164-1212(02)00156-5

Langseth, H. & Portinale, L. (2007). Bayesian Networks in Reliability. *Reliability Engineering & System Safety*, 92(1), 92-108. doi: 10.1016/j.ress.2005.11.037

Lyngdog, G., Rahman, M., & Mishra, S. (2019). Bayesian Updating of Structural Model with a Conditionally Heteroscedastic Error Distribution. *Journal of Engineering Mechanics*, 145(12), 1-18. doi: 10.1061/(ASCE)EM.1943-7889.0001668

Marquez, D., Neil, M., & Fenton, N. (2010). Improved reliability modeling using Bayesian networks and dynamic discretization. Reliability Engineering and System Safety, 95(4), 412–425. doi: 10.1016/j.ress.2009.11.012

Namazian, A. & Haji Yakhchali, S. (2018). Developing an Integrated Simulation Model of Bayesian-networks to Estimate the Completion Cost of a Project under Risk: Case Study on Phase 13 of South Pars Gas Field Development Projects. *Industrial Management Journal*, 10(3), 407-428. doi: 10.22059/imj.2018.228669.1007202

O'Hagan, A. (2019). Expert Knowledge Elicitation: Subjective but Scientific. *The American Statistician*. 73(sup1), 69-81. doi: 10.1080/00031305.2018.1518265

Pollino, C. A., Woodberry, O., Nicholson, A., Korb, K., & Hart, B. T. (2007). Parameterisation and evaluation of a Bayesian network for use in an ecological risk assessment. Environmental Modelling and Software, 22(8), 1140–1152. doi: 10.1016/j.envsoft.2006.03.006

Su, A., Fan, M., & Li, Z. (2012). The reliability analysis of distribution system based on Dynamic Bayesian Network. 2012 China International Conference on Electricity Distribution, Electricity Distribution (CICED), 2012 China International Conference On, 1–4. doi: 10.1109/CICED.2012.6508483

Sun, Z., Hupman, A., Ritchey, H., & Abbas, A. (2016). Bayesian Updating of the Price Elasticity of Uncertain Demand. *IEEE Systems Journal*, 10(1), 136-146. doi: 10.1109/JSYST.2014.2315192

Thomas, D. & Fitch, J. (2014). Joint Agency Cost Schedule Risk and Uncertainty Handbook. https://www.ncca.navy.mil/tools/csruh/JA_CSRUH_16Sep2014.pdf

Vialett, E., Humbert, N., & Mottier, P. (2019). Updating a Probabilistic Seismic Hazard Assessment with Instrumental and Historical Observations Based on a Bayesian Inference. *Nuclear Engineering and Design*, (350), 98-106. doi: 10.1016/j.nucengdes.2019.04.034

Whyte, G., & Classen, S. (2012). Using storytelling to elicit tacit knowledge from SMEs. *Journal of Knowledge Management*, *16*(6), 950-962. doi: http://dx.doi.org/10.1108/13673271211276218

Conclusion of Articles

Article one finds a method of moments utilizing mean and variance of a given triangular distribution provides the best technique for estimating the parameters of a beta distribution when considering empirical fit and simplicity. The difference between both individual distributions and when aggregating multiple distributions (as evidenced by the simulated empirical example) is marginal. Furthermore, the fit provided using the combination of mean and variance provides a fit very similar to the more rigorous method of quantile estimation with the added advantage of being implementable by a junior level analyst with a minimum level of mathematical understanding.

Article two applies the findings of its predecessor to provide a technique to serve as a framework to account for underestimation in cost models as well provide an alternative method for combining subject matter expert opinions. In most cases, using Bayesian updating resulted in increasing the budget for subjectively estimated cost elements by at least 40%. For the combination of expert opinion, characterizing the Bayesian updated distribution's stochastic dominance versus the Coleman et al. method appears to better fit the narrative given the scenario.

In both of these instances it is important to note that there are alternative methods that could be used. For the budgeting application, subjective distributions used for budget estimates could simply be updated by the expert or the mode could be moved to match the data point as examples. However, the Bayesian framework takes a considerable degree of uncertainty out of these potential updating methods and only asks that a decision maker define a data-generating function for the updating process.

In the application of combining expert opinion, one could use a weighted average of the parameters in order to take into consideration the fact that expert two has some additional knowledge informing their distribution. However, this method again adds uncertainty in the sense that one must provide another piece of subjectivity in determining the weights of each expert's parameters.

There is a multitude of future research to be highlighted in closing -- mostly focused around using empirics to quantify the characterization of these manuscripts. For paper 1, it is recommended that a dataset consisting of old estimates with triangular distributions used for uncertainty be gathered along with actual costs for the corresponding estimates. This would allow a researcher to quantify whether the beta distribution better predicts actual costs than the given triangular distribution. Additionally, this research could be expanded to model the four-parameter beta distribution in order to accurately account for variance in cost between different cost elements. Finally, additional methods of estimating the parameters for the beta distribution could be explored to include MLE and further study on quantile estimation.

For paper 2, future research should focus on gather data of percent of programs in the DoD with cost overruns in order to quantify the amount of error the Bayesian model empirically accounts for. Further analysis could be done to not only weight the percentage of programs that experience cost overruns, but also to weight the magnitudes of said overruns. Finally, work could be done to gather actual instances where multiple experts provide inputs and combine their distributions. These combined distributions could be compared to actual costs to determine if Coleman et al., a Bayesian framework, or a weighted version of Coleman et al. more accurately predicts costs.

REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 074-0188*

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED** *(From – To)*
26-03-2020	Master's Thesis	August 2018 - March 2020

TITLE AND SUBTITLE Analysis of Beta Distribution for Subjective Uncertainty in Cost Models	**5a. CONTRACT NUMBER**
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S) Stafford, Ryan D., 1Lt, USAF	**5d. PROJECT NUMBER**
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(S) Air Force Institute of Technology Graduate School of Engineering and Management (AFIT/EN) 2950 Hobson Way, Building 640 WPAFB OH 45433-8865	**8. PERFORMING ORGANIZATION REPORT NUMBER** AFIT-ENV-MS-20-M-241
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) Intentionally left blank	**10. SPONSOR/MONITOR'S ACRONYM(S)**
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

Subjective uncertainty exists within the realm of cost estimation. Typical methodology for subjective uncertainty involves elicitation from a subject matter expert to provide a high, low, and most likely value -- defining a triangular distribution -- to model said uncertainty. This manuscript explores ways to leverage research on elicitation geared towards defining a triangular distribution and provide a simple conversion to a beta distribution usable by cost analysts with various degrees of mathematical knowledge. Furthermore, this manuscript attempts to demonstrate the benefits of using a beta distribution through its application as a conjugate prior for Bayesian updating in cost models.

15. SUBJECT TERMS

Triangular Distribution, Beta distribution, Bayesian updating, subjective uncertainty, subject matter expert

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON** Koschnick, Clay M., Lt Col, USAF ADVISOR
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**			**19b. TELEPHONE NUMBER** *(Include area code)*
U	U	U	UU	46	(937) 255-3636, ext 4638 (clay.koschnick@afit.edu)

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39-18