Focused Beam System Biaxial Material Characterization

Nicholas A. O’Gorman

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Focused Beam System Biaxial Material Characterization

THESIS

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AFIT-ENG-MS-20-M-050

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

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Focused Beam System Biaxial Material Characterization

THESIS

presented to the Faculty
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Graduate School of Engineering and Management
Air Force Institute of Technology
Air University
Air Education and Training Command
In Partial Fulfillment of the Requirements for the
Degree of Master of Science in Electrical Engineering

Nicholas A. O’Gorman, BS
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March 2020

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Focused Beam System Biaxial Material Characterization

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Nicholas A. O’Gorman
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Abstract

Electromagnetic material characterization is the process of determining the constitutive parameters (complex permittivity and permeability) of a given sample. Due to the large number of unknowns involved, multiple unique measurements are required for material property extraction. Many measurement methods, such as waveguides and striplines, possess a rigid internal structure that the sample being measured must adhere to. This rigidity limits these methods to samples that fit within the device and inhibits oblique sample orientations, limiting the number of independent measurements that can be obtained. A focus beam system, due to being an open system with greater freedom in sample size and orientation, can collect a larger number of measurements that could not be obtained otherwise. With this greater number of unique data sets, the focus beam system can solve for the material properties of more complex materials that could not be achieved using other methods.

In this thesis paper, a method for extracting the material parameters for a biaxial material using a focus beam system is derived and tested, including for the case when the orientation of the sample’s axes are unknown. The results are then compared to those obtained utilizing other methods with the same material design, verifying that the methodologies work.
1. Introduction

With the rise of 3D printers and other cheap production methods for unique and cheap materials, the possibilities in material design and associated costs have completely changed. This boom in possibilities has also affected the world of electronics, allowing the production of materials that can interact with the electromagnetic spectrum in new and complex ways. Such complexity has also given rise to a need to test and verify that these new materials are able to operate in the manner for which they have been designed.

Before, when limited to mostly isotropic samples, many methodologies existed to test a material’s electromagnetic properties. However, as the level of complexity increases, many of these methods are no longer able to easily obtain solutions to the growing number of unknowns. This is mostly due to the constraints of many of the methods used. Samples need to be of a certain size and orientation for accurate measurements to be obtained through such devices as waveguides and coax systems. One method that does not possess such a rigid setup is the Focus Beam System (FBS).

The FBS sets itself apart from other methods due to its lack of a closed system. This means that a sample only needs to reside between the two antennas instead of having to be pressed between a waveguide or some other device used to keep the electromagnetic waves in a closed system. Removing this restriction allows many differently sized materials to fit within the system for testing. A FBS also does not require the use of harmonics based on the size of the closed system, greatly increasing the frequencies that the sample can be tested. Lastly, since the sample is suspended in free space, it can be easily orientated at different oblique angles, obtaining a larger number of independent measurements.

One of the major limitations of the FBS is the gaussian beam depth of the focus region where the field can be considered planar. This problem gets further increased when oblique incidence is introduced as it offsets the place field when it is focused towards the antenna. Another limitation of the FBS is based on the antennas/lenses and the diameter of the sample. The antenna/lenses need to be designed to operate at the intended frequencies that one wishes to test for accurate results as the desired plane wave will become less of an accurate representation when one deviates further from the designed parameters.
1.1 Problem Statement

Current methods utilizing a FBS possess a couple limitations. The first one being that both the axis of the sample being tested, and the axis of the antennas need to be aligned to obtain accurate information. This means that if the axis orientation of the material parameters for a sample, such as biaxial or uniaxial, is not known, then current methods will be unable to get accurate results. A second limitation of current methods is that it is restricted to two of the three axes of the material. When the sample is in line with the antennas, assuming the Z axis represents the direction of propagation between the antennas, then it is only possible to obtain the material parameters correlating with the X and Y axes since the H and E field cannot point towards the direction of travel inside of a plane wave. Most samples are then unable to be rotated so that the Z axis of the sample now lines up with one of the measurable axes as one axis of the samples tend to be too thin to collect accurate data without fringing effects distorting the data.

Overcoming the need for a known axis orientation is important because the sample orientation might not always be discernible. If it is a simple material that was produced by those doing the measurements, then the axes are generally known. However, if one is testing a sample with a more complex physical design or if one is testing a part that they did not personally produce, knowing the axes orientation becomes more challenging.

Not being able to obtain results for the Z axis of the sample can be very problematic. If the sample is a uniaxial or isotropic material, then this limitation does not pose a problem. If the material is a biaxial material where all three axes are required to have certain values, one is unable to verify that the third axis resides within acceptable error margins.

The goal of this thesis is to derive and test a methodology that allows for the determination of the Z axis material properties by rotating the sample so it is no longer perpendicular to the antennas, allowing the Z axis of the material parameters to enter the path of the emitted plane wave. Next, a method for extracting all three diagonal material parameters for an unknown axis orientation is investigated. Then lastly, a way to combine both methods together is discussed. This will be done by utilizing cross polarization measurements taken with a FBS in conjunction with the properties inherent to biaxial materials.
1.2 Limitations

The limitations of the methods described in this paper are based on a few assumptions that are used in the calculations. It is assumed that the electromagnetic waves passing through the sample are plane waves. For this to be valid, the FBS lenses must transform the near field EM waves from the antenna into the far field. The sample must also possess a flat uniform surface that the electromagnetic waves can enter and leave without experiencing varying thicknesses at different sections of the sample. The material parameters must also remain constant for each axis of the sample being tested or the results will be an average based on what portions the waves interacted with.

1.3 Scope

This thesis focuses only on the FBS and its flexibility in sample orientation to extract material parameters. The methods derived will examine both $S_{21}$ and $S_{11}$ values. Since the samples tested are non-magnetic, only the $S_{21}$ parameters will be used during the experimentation. Due to a lack of large samples to test, the frequency ranges used will reside within the 15 to 18 GHz range for the electromagnetic waves to only pass through the area that the biaxial sample will reside in. All the methods derived assume that the sample being tested only possesses material parameters along the diagonal of both the permittivity and permeability tensor.

1.4 Thesis Organization

Chapter 2 provides a theoretical background for the basic electromagnetic and physics concepts that are needed to understand the methods utilized. Chapter 3 derives the equations used to extract the material parameters for the sample. Chapter 4 discusses the methodology of the experiment conducted. Chapter 5 derives the error associated with the measurements for this methodology. Chapter 6 presents the experimental results and discusses what they represent. Lastly, a conclusion with recommendations for future research is presented in Chapter 7.
2. Background Information

To obtain an understanding of the what and why behind this research, it is necessary to obtain a fundamental understanding of the underlying theories that it stems from. This section focuses on giving an overview of the theory behind this research project and the previous work that has led to it.

2.1 Electromagnetic Fields

Electromagnetic (EM) theory is the branch of scientific research and technology that involves utilizing, generating, and interacting with electromagnetic waves, both artificial and natural, that exist all around us. These waves represent everything from the visible spectrum that we use to see, to solar radiation that provides the energy for all life on Earth [2]. Figure 2.1 shows how the different frequencies of EM waves can be experienced in everyday life.

![Figure 2.1. The Electromagnetic Spectrum [1].](image)

EM waves can be represented as the movement of energy at the speed of light. The amount of energy that is contained in a wave is directly proportional to the amplitude of the field and can be generated using time-varying currents. All time-varying magnetic fields induce an electric field and all time-varying electric fields induce a magnetic field with each always being perpendicular to one another in simple media. Figure 2.2 shows this direct relationship with an EM field traveling along
2.2 Maxwell’s Equations

All EM fields follow a set of rules represented by Maxwell’s equations. These equations were compiled and simplified into their current form by Heaviside. [2]. The differential form of Maxwell’s equations can be written as follows:

\[ \nabla \times E = -J_m - \frac{\partial B}{\partial t} \] (2.1)

Equation 2.1 is known as Faraday’s Law and dictates that the curl (\( \nabla \times \)) of the electric field (\( E \)) is equivalent to the negative magnetic current (\( J_m \)) minus the partial time derivative (\( \frac{\partial}{\partial t} \)) of the magnetic flux density (\( B \)).

\[ \nabla \times H = J_e + \frac{\partial D}{\partial t} \] (2.2)
Equation 2.2 is known as Ampere’s law and dictates that the curl of the magnetic field \( \mathbf{H} \) is equivalent to the electric current \( J_e \) plus the partial time derivative of the electric flux density \( \mathbf{D} \).

\[
\nabla \cdot \mathbf{D} = \rho_e \tag{2.3}
\]

\[
\nabla \cdot \mathbf{B} = \rho_m \tag{2.4}
\]

Equation 2.3 and 2.4 are both referred to as Gauss’s law. Equation 2.3 dictates that the divergence \( (\nabla \cdot) \) of the electric flux density is equal to the electric charge density \( \rho_e \). Equation 2.4 dictates that the divergence of the magnetic flux density is equal to the magnetic charge density \( \rho_m \).

Although neither magnetic currents nor magnetic charge densities have been observed, they are kept so that the above equations remain symmetric and are used in field equivalence problems to aid in calculations.

The fields and their corresponding fluxes also have a mathematical relationship that is depicted in the following equations.

\[
\mathbf{B}(\mathbf{r}, \omega) = \bar{\mu}(\omega) \cdot \mathbf{H}(\mathbf{r}, \omega) \tag{2.5}
\]

Equation 2.5 dictates that the magnetic flux is equal to the permeability \( \bar{\mu} \) tensor of the medium the wave is traveling through multiplied by the magnetic field.

\[
\mathbf{D}(\mathbf{r}, \omega) = \bar{\varepsilon}(\omega) \cdot \mathbf{E}(\mathbf{r}, \omega) \tag{2.6}
\]

Equation 2.6 dictates that the electric flux is equal to the permittivity \( \bar{\varepsilon} \) tensor of the medium the wave is traveling through multiplied by the electric field.

2.3 Material Properties

All materials are made of atoms. Atoms are around \( 10^{-10} \) meters in size and a very small and dense nucleus made up of uncharged neutrons and positively charge protons. The nucleus of an atom is surrounded by smaller, negatively charged electrons referred to as the electron cloud, depicted in Figure 2.3. The number of protons determines what type of substance it is. For
example, Figure 2.3 depicts Germanium as it has 32 protons. The amount of energy it takes to remove an electron from the electron cloud is the main factor in determining the electrical properties of that substance [2].

![Germanium Atom Depicted with Electron Shell](image)

**Figure 2.3. Germanium Atom Depicted with Electron Shell [2].**

### 2.3.1 Permittivity

Different materials exhibit different reactions to magnetic and electric fields based on the atoms and atomic bonds that they have with one another. One type of material that will be discussed are classified as dielectrics. The defining characteristic of a dielectric is a material whose electrons and nucleus are both held in place by atomic and molecular forces. This means that the electrons are not free to move around the material like they would be able to in a conductor.

Though the atoms are not capable of moving within the material, an electric field can generate a force on the electrons and protons, shifting their relative positions in the atom. The atom normally exists in the orientation depicted in Figure 2.4a with the nucleus centered and the electron revolving around it. However, once an electric field is applied, the force that is induced acts in opposite directions for the oppositely charged nucleus and electrons. This displaces the electron cloud from the nuclear center (Figure 2.4b) creating what is known as an orientational...
polarization which can be represented by a dipole (Figure 2.4c) described by the partial dipole equation (Equation 2.7).

\[ dp_i = Q\ell_i \]  

Equation 2.7 dictates that the partial electric dipole \( dp_i \) is equal to the charge \( Q \) multiplied by the distance between the charges \( \ell_i \).

Normally it is more useful to represent the dipole moment as the sum for a unit volume, instead of atom by atom. This is done by adding together all the partial dipoles in a unit area, leading to the following.

\[ P = N_e Q\ell_{av} \]  

Equation 2.8 dictates that the dipole moment per unit volume \( P \) is equal to the number of dipole moments in the unit volume \( N_e \) multiplied by the dipole charge \( Q \) and the average distance length of the dipole \( \ell_{av} \).

The macro representation of the electric polarization produces a vector pointing in the same direction as the electric field for isotropic materials as can be seen in Figure 2.5.

Electric flux density for free space is defined by

\[ D_0 = \epsilon_0 E \]  

where electric flux density of free space \( D_0 \) equals the permittivity of free space \( \epsilon_0 \) multiplied with the electric field. When a simple dielectric material is subjected to an electric field, this
relationship changes due to the induced polarization resulting in the electric flux equal to the permittivity of free space multiplied by the electric field plus the total dipole.

\[ \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (2.10) \]

Equation 2.10 can then be simplified to the electric flux equals the permittivity of the medium (\(\bar{\epsilon}(\omega)\)) multiplied by the electric field.

\[ \mathbf{D}(\mathbf{r}, \omega) = \bar{\epsilon}(\omega) \mathbf{E}(\mathbf{r}, \omega) \quad (2.11) \]

where permittivity is defined as the relative permittivity (\(\bar{\epsilon}_r(\omega)\)) multiplied by the permittivity of free space.

\[ \bar{\epsilon}(\omega) = \bar{\epsilon}_r(\omega) \epsilon_0 \quad (2.12) \]

### 2.3.2 Permeability

Similar to dielectric materials generating electric polarization when subject to an electric field, a magnetic material will generate magnetic polarization when subject to a magnetic field. To demonstrate this, an atom can be represented with equivalent current loops instead of electrons orbiting the nucleus as seen in Figure 2.6a. These current loops produce a magnetic field as
Figure 2.6. Electric Current Loop Equivalents [2].

d_m = \hat{n}_i I_i ds_i \tag{2.13}

Equation 2.13 dictates that the partial magnetic dipole moment ($d_m$) is equal to the normal vector of the current loop ($\hat{n}_i$) times the loop current ($I_i$) and the surface area of the current loop ($ds_i$).

It is then possible to add all the magnetic dipole moments for each electron per volume to calculate the magnetic polarization vector $M$ shown in Equation 2.14.

\[ M = \hat{n} N_m (I ds_i)_{av} \tag{2.14} \]

Equation 2.14 dictates that the Magnetic polarization vector ($M$) is equal to the normal vector to the current loop multiplied by the number of magnetic dipole moments ($N_m$) and the average current multiplied by the current loop surface area (($I ds_i)_{av}$).

As the magnetic field passes through the material, a torque is produced on the magnetic dipoles. The torque rotates the dipoles towards the same direction as the magnetic field as shown in Figure 2.7. The alignment of magnetic dipoles increases the magnetic flux density in the material the same way that the electric polarization vector did for electric flux density. The
relationship between magnetic polarization and magnetic flux density for simple media is as follows:

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$$  \hspace{1cm} (2.15)$$

with the magnetic flux density equal to the permeability of free space ($\mu_0$) multiplied by the magnetic field and magnetic polarization added together.

The magnetic flux density can also be represented using Equation 2.16 where magnetic flux density equals the permeability of the material ($\mu_r(\omega)$) multiplied by the magnetic field,

$$\mathbf{B}(\mathbf{r}, \omega) = \mu_r(\omega) \cdot \mathbf{H}(\mathbf{r}, \omega)$$  \hspace{1cm} (2.16)$$

where permeability is defined as the relative permeability ($\mu_r(\omega)$) multiplied by the permeability of free space.

$$\mu_r(\omega) = \mu_r(\omega)\mu_0$$  \hspace{1cm} (2.17)$$
2.4 Metamaterials

Metamaterials are defined as an artificially created material that exhibits properties that go beyond what can be found in nature [2]. For this research effort, only the EM properties of metamaterials will be investigated. The interactions between a simple media and an EM field is dictated by three constitutive parameters; permittivity ($\varepsilon$), permeability ($\mu$) and conductivity ($\sigma$). This research focuses on the permittivity and permeability material properties.

In EM, the simplest materials are isotropic. Isotropic materials’ interactions with EM fields are independent from direction, meaning that no matter the orientation of the material, the EM fields will perceive the exact same permittivity and permeability. This results in the material parameters represented in Equation 2.18, where only the diagonals of the tensors are non-zero and all three possess the same value.

$$\mathbf{\varepsilon} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_1 \end{pmatrix}, \quad \mathbf{\mu} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix} \quad (2.18)$$

A special case for isotropic materials is air. The material parameters for air can be approximated as free space, giving it the following material properties:

$$\mathbf{\varepsilon} = \begin{pmatrix} \epsilon_0 & 0 & 0 \\ 0 & \epsilon_0 & 0 \\ 0 & 0 & \epsilon_0 \end{pmatrix}, \quad \mathbf{\mu} = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_0 & 0 \\ 0 & 0 & \mu_0 \end{pmatrix} \quad (2.19)$$

Another type of material are biaxial materials. Biaxial materials do not possess directional independence and can interact with EM fields differently depending on the direction the waves come from. The resulting material parameters are represented in Equation 2.20, possessing only non-zero values on the principle axis but are no longer required to be equal.

$$\mathbf{\varepsilon} = \begin{pmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}, \quad \mathbf{\mu} = \begin{pmatrix} \mu_{xx} & 0 & 0 \\ 0 & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{pmatrix} \quad (2.20)$$

A third type of material classification has both directional dependency and cross-polarization terms associated with it (non principle axis). These types of material are called anisotropic.
anisotropic material has the potential to excite fields along different axes than the incoming wave’s axis. For example, an \( \hat{x} \) directed field entering an anisotropic material could emit a \( \hat{x} \) and \( \hat{y} \) directed field. Anisotropic material properties are depicted in Equation 2.21.

\[
\bar{\varepsilon} = \begin{pmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{pmatrix}
\quad \bar{\mu} = \begin{pmatrix}
\mu_{xx} & \mu_{xy} & \mu_{xz} \\
\mu_{yx} & \mu_{yy} & \mu_{yz} \\
\mu_{zx} & \mu_{zy} & \mu_{zz}
\end{pmatrix}
\] (2.21)

### 2.5 S Parameters

S parameters are a measure of the relationship between the emitted and received electromagnetic waves and are used in material analysis. These parameters can be collected by using a two-port network analyzer such as the performance network analyzer (PNA) utilized for this research. A basic image of how a two-port network analyzer will emit and receive EM waves is displayed in Figure 2.8. \( a_1 \) is the generated signal from port 1, \( a_2 \) is the generated signal from port 2, \( b_1 \) is the received signal of port 1, and \( b_2 \) is the received signal of port 2 [5].

\[
\begin{align*}
& a_1 \quad \longrightarrow \quad \boxed{\quad} \quad \longleftrightarrow \quad \quad a_2 \\
& b_1 \quad \longleftrightarrow \quad \boxed{\quad} \quad \longrightarrow \quad b_2
\end{align*}
\]

**Figure 2.8. Linear 2 Port Network Analyzer S Parameters**

From these variables, the S parameters are defined as shown in Equation 2.22.

\[
\begin{align*}
S_{11} &= \frac{b_1}{a_1} \bigg|_{a_2=0} \\
S_{12} &= \frac{b_1}{a_2} \bigg|_{a_1=0} \\
S_{21} &= \frac{b_2}{a_1} \bigg|_{a_2=0} \\
S_{22} &= \frac{b_2}{a_2} \bigg|_{a_1=0}
\end{align*}
\] (2.22a-d)

Utilizing these definitions, it is possible to generate an equation for the received signal in port
one shown in Equation 2.23a and the received signal of port two in Equation 2.23b.

\[ b_1 = S_{11}a_1 + S_{12}a_2 \]  
(2.23a)

\[ b_2 = S_{21}a_1 + S_{22}a_2 \]  
(2.23b)

The equations can then be utilized in a matrix depicting the relationship between the emitted and received signals using S parameters as shown in Equation 2.24.

\[
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{21} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}
\]  
(2.24)

### 2.6 A Matrix

Aside from S parameters, another method for analyzing a material is using an A matrix. The difference between S and A matrices is that an A matrix has one side used for inputs and the other side for outputs as shown in Figure 2.9 [5]. This method allows the A matrix to follow the same pattern as a group of samples stacked together. For example, sample 1 held to the left of sample 2 is equal to \( A_1A_2 \).

\[
a_1 \rightarrow \begin{array}{c} \text{Sample 1} \\ \text{Sample 2} \end{array} \rightarrow a_2
\]

\[
b_1 \rightarrow \begin{array}{c} \text{Sample 1} \\ \text{Sample 2} \end{array} \rightarrow b_2
\]

Figure 2.9. 2 Port Network Analyzer A Parameters

The resulting equations that are formed from the setup depicted in Figure 2.9 is as follows:

\[ a_1 = A_{11}b_2 + A_{12}a_2 \]  
(2.25a)

\[ b_1 = A_{21}b_2 + A_{22}a_2 \]  
(2.25b)

These can then be combined into the following matrix.
\[
\begin{pmatrix} a_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} b_2 \\ a_2 \end{pmatrix}
\] (2.26)

A relationship can then be found between the A matrix and S matrix to transform the collected S parameters to A parameters and back.

Equation 2.23b can be rearranged to show the following:

\[ a_1 = \frac{1}{S_{21}} b_2 - \frac{S_{22}}{S_{21}} a_2 \] (2.27)

When Equation 2.27 is compared to Equation 2.25b, the relationship for \( A_{11} \) and \( A_{12} \) is obtained.

\[ A_{11} = \frac{1}{S_{21}}, \quad A_{12} = -\frac{S_{22}}{S_{21}} \] (2.28)

Next, Equation 2.27 can be Incorporated into Equation 2.23a and rearranged to give:

\[ b_1 = \frac{S_{11}}{S_{21}} b_2 + \frac{S_{12} S_{21} - S_{11} S_{22}}{S_{21}} a_2 \] (2.29)

Comparing Equation 2.29 and Equation 2.25a, the relationship between \( A_{21} \) and \( A_{22} \) is obtained.

\[ A_{21} = \frac{S_{11}}{S_{21}}, \quad A_{22} = \frac{S_{12} S_{21} - S_{11} S_{22}}{S_{21}} \] (2.30)

Combining Equation 2.30 and Equation 2.28 the S to A transformation is found.

\[ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \frac{1}{S_{21}} \begin{pmatrix} 1 & -S_{22} \\ S_{11} & S_{12} S_{21} - S_{11} S_{22} \end{pmatrix} \] (2.31)

The relationship to transform A parameters back into S parameters can also be found by rearranging Equation 2.25b to solve for \( b_2 \).

\[ b_2 = -\frac{A_{12}}{A_{11}} a_2 + \frac{1}{A_{11}} a_1 \] (2.32)

Comparing Equation 2.32 and Equation 2.23a, \( S_{22} \) and \( S_{21} \) is obtained.

\[ S_{22} = -\frac{A_{12}}{A_{11}}, \quad S_{21} = \frac{1}{A_{11}} \] (2.33)
Equation 2.33 can then be incorporated into Equation 2.25a and rearranged, giving

\[ b_1 = \frac{A_{21}}{A_{11}}a_1 + \frac{A_{11}A_{22} - A_{21}A_{12}}{A_{11}}a_2 \]  
(2.34)

Equation 2.34 can then be compared to Equation 2.23a to produce the following relationship for \( S_{11} \) and \( S_{21} \).

\[ S_{11} = \frac{A_{21}}{A_{11}}, \quad S_{21} = \frac{A_{11}A_{22} - A_{21}A_{12}}{A_{11}} \]  
(2.35)

Equation 2.35 and Equation 2.33 can then be combined to give the \( A \) to \( S \) matrix transformation.

\[
\begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix}
= \frac{1}{A_{11}}
\begin{pmatrix}
A_{21} & A_{11}A_{22} - A_{21}A_{12} \\
1 & -A_{12}
\end{pmatrix}
\]  
(2.36)

### 2.7 X-Y Cross Polarized S Parameters

When a material possesses \( \epsilon_{xy/yx} \) or \( \mu_{xy/yx} \) then cross polarization exists between the \( \hat{x} \) and \( \hat{y} \) directed \( \vec{E} \) and \( \vec{H} \) fields. For example, when an \( \hat{x} \) directed \( \vec{E} \) field intercepts with the sample, then the reflected and transmitted \( \vec{E} \) fields will be in both the \( \hat{x} \) and the \( \hat{y} \) direction, generating more complex S parameters as shown in Figure 2.10.

These new variables cause the S parameters to become block 2x2 matrices for all the possible combinations of \( \hat{x} \) and \( \hat{y} \). The S parameters are then defined as follows:
Equation 2.37 can then be organized into the relationships defining each input

$$S_{11}^{xx} = \frac{b_1^x}{a_1^x} |a_{1y}^x, a_{2y}^x, a_{2y}^x = 0$$  \hspace{1cm} (2.37a)

$$S_{12}^{xx} = \frac{b_1^x}{a_1^x} |a_{1y}^y, a_{2y}^y, a_{2y}^y = 0$$  \hspace{1cm} (2.37b)

$$S_{11}^{yy} = \frac{b_1^y}{a_1^y} |a_{1y}^x, a_{2y}^x, a_{2y}^x = 0$$  \hspace{1cm} (2.37c)

$$S_{12}^{yy} = \frac{b_1^y}{a_1^y} |a_{1y}^y, a_{2y}^y, a_{2y}^y = 0$$  \hspace{1cm} (2.37d)

$$S_{21}^{xx} = \frac{b_2^x}{a_2^x} |a_{1y}^x, a_{2y}^x, a_{2y}^x = 0$$  \hspace{1cm} (2.37e)

$$S_{22}^{xx} = \frac{b_2^x}{a_2^x} |a_{1y}^y, a_{2y}^y, a_{2y}^y = 0$$  \hspace{1cm} (2.37f)

$$S_{21}^{yy} = \frac{b_2^y}{a_2^y} |a_{1y}^x, a_{2y}^x, a_{2y}^x = 0$$  \hspace{1cm} (2.37g)

$$S_{22}^{yy} = \frac{b_2^y}{a_2^y} |a_{1y}^y, a_{2y}^y, a_{2y}^y = 0$$  \hspace{1cm} (2.37h)

that can then be combined into a matrix.

$$\begin{pmatrix}
  b_1^x \\
  b_1^y \\
  b_2^x \\
  b_2^y
\end{pmatrix} = 
\begin{pmatrix}
  S_{11}^{xx} & S_{11}^{xy} & S_{12}^{xx} & S_{12}^{xy} \\
  S_{11}^{yx} & S_{11}^{yy} & S_{12}^{yx} & S_{12}^{yy} \\
  S_{21}^{xx} & S_{21}^{xy} & S_{22}^{xx} & S_{22}^{xy} \\
  S_{21}^{yx} & S_{21}^{yy} & S_{22}^{yx} & S_{22}^{yy}
\end{pmatrix}
\begin{pmatrix}
  a_1^x \\
  a_1^y \\
  a_2^x \\
  a_2^y
\end{pmatrix}$$  \hspace{1cm} (2.39)
2.8 Focus Beam System

A FBS is a device that is used to analyze electromagnetic material parameters. It is composed of two antennas, two lenses, a device to hold the sample material in place, and a network analyzer (PNA) as depicted in Figure 2.11.

A FBS works by sending an EM wave out from Antenna 1, the wave then interacts with the first lens, transforming the wave into a plane wave. The wave then intercepts the material and is affected by the sample’s material parameters. This interaction causes a portion of the wave to be reflected towards Antenna 1 and another portion to travel through the material. The transmitted portion of the wave then intercepts the second lens and is focused towards the second antenna. The second antenna then receives the signal and sends the information to the port analyzer. The reflected wave follows a similar process but is received by the first antenna. The same process is followed with a signal sent from antenna 2.

The benefit of working with a FBS is that the sample size and shape are only limited by the range of the usable frequencies of the FBS setup. This limitation is to remain within the operating frequencies of both the antennas and lenses. The FBS, due to its use of focal lenses, allows for calculations to be done in the far-field, simplifying the equations and making the data retrieved easier to process [5]. Another benefit of the FBS is the ability to collect more unique sets of measurements than a wave guide due to the ability to rotate samples around different axes. The increased number of unique measurements allow for solutions of metamaterials with higher levels of complexity (more unknowns) to be obtained.
2.9 Gating

For a FBS, the reflection and transmitted values collected include all EM waves reflected from all surfaces in the room and any other EM waves that might be propagating in the room during data collection. There also exists internal reflections inside of the coaxial cables between the PNA and the FBS. The only desired data is the portion of the waves that went directly through the sample to the opposite antenna and the waves that reflected directly off the sample and returned to the original antenna. Since this is the most direct route for the EM waves to travel, they experience the least amount of dispersion when compared with waves that travel a less direct route. Using this knowledge, it is possible to isolate the direct path by finding the maximum value in the time domain and then diminish the incoming waves that reside outside of this maximum [6].

The S parameters collected from the PNA are in the frequency domain, so the first step is to take the inverse Fourier transformation of the data to switch to the time domain. The frequencies that are collected exist as a discrete finite number of samples, making the discrete inverse Fourier transformation needed, shown in Equation 2.40 [7].

\[
x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[K] e^{jK\omega n}
\]  

(2.40)

\(N\) represents the total number of samples, \(K\) is the data location in the frequency domain, \(n\) is the location in the time domain data set, and \(\omega\) is the angular frequency.

With the time domain of the S parameters now obtained, the maximum value in the set must then be found. Once located, shift the entire data set so that the maximum value in the data set resides along the 0 position. The total data set is then multiplied by a sinc function to remove the noise around the maximum value. The resulting values are then shifted back to their original position and Fourier transformed back into the frequency domain using Equation 2.41.

\[
X[K] = \sum_{n=0}^{N-1} x[n] e^{-jK\omega n}
\]  

(2.41)

The resulting S parameters now have the noise in the data mitigated and the direct path remaining.

2.10 Isolate Biaxial Sample S Parameters

Once the noise in the data set is removed, the S parameters now represent the entire space between the two antennas. The S parameters of only the sample that is being tested are needed.
Therefore, the S parameters pertaining to the air between the antennas and sample needs to be removed [5]. Figure 2.12 shows what resides in the gated S parameters with Section A being the space between antenna 1 and the sample, Section B being the space between the sample and antenna 2, and Sample representing the biaxial material.

<table>
<thead>
<tr>
<th>Section A</th>
<th>Sample</th>
<th>Section B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^A$</td>
<td>$S^S$</td>
<td>$S^B$</td>
</tr>
</tbody>
</table>

Figure 2.12. Gated S Parameters

Using Figure 2.12, it can be shown that the Gated S parameters are equal to the following:

\[
S^M_{21} = S^A_{21}S^S_{21}S^B_{21} \quad S^M_{12} = S^B_{12}S^S_{12}S^A_{12}
\]  
\[
S^M_{11} = S^A_{21}S^S_{11}S^A_{12} \quad S^M_{22} = S^B_{12}S^S_{22}S^B_{21}
\]  

(2.42a)  
(2.42b)

with $S^M$ being the measured sample after being gated, $S^S$ equal to the sample S parameters, and $S^A$, $S^B$ representing the S parameters for Section A and Section B respectively.

Equation 2.42a and Equation 2.42b can be rearranged to solve for the sample’s S parameters.

\[
S^S_{21} = \frac{S^M_{21}}{S^A_{21}S^B_{21}} \quad S^S_{12} = \frac{S^M_{12}}{S^B_{12}S^A_{12}}
\]  
\[
S^S_{11} = \frac{S^M_{11}}{S^A_{21}S^A_{12}} \quad S^S_{22} = \frac{S^M_{22}}{S^B_{12}S^B_{21}}
\]  

(2.43a)  
(2.43b)

The S parameters for $S^A$ and $S^B$ will need to be removed as they are unknown variables. A second set of equations will need to be obtained by measuring the S parameters for the system with the sample removed resulting in the following:

\[
S^M_{21} = S^A_{21}e^{-jK_0d}S^B_{21} \quad S^M_{12} = S^B_{12}e^{-jK_0d}S^A_{12}
\]  

(2.44)

where $S^M$ is the measured through S parameter with the sample removed and $e^{-jK_0d}$ is the equivalent phase shift in free space across the thickness of the sample with $K_0$ being the propagation constant in free space.
Taking Equation 2.44 and Equation 2.43a the following relationship can be found

\[ S_{21}^S = \frac{S_{MS}^{11}}{S_{MT}^{21}} e^{-jK_0d} \quad S_{12}^S = \frac{S_{MS}^{12}}{S_{MT}^{21}} e^{-jK_0d} \] (2.45)

Equation 2.45 allows for \( S_{21}^S \) and \( S_{12}^S \) to be isolated. \( S_{11}^S \) and \( S_{22}^S \) however need to be solved differently since the unknowns in Equation 2.43b do not directly relate to the unknowns for the through measurement. One method is to combine \( S_{11}^S \) and \( S_{22}^S \) so that the unknowns in the equation match the unknowns in the through measurements \[8\].

\[ S_{11}^S S_{22}^S = \frac{S_{MS}^{11} S_{MS}^{22}}{S_{21}^S S_{21}^S S_{12}^S S_{12}^S} \] (2.46)

Using Equation 2.46 and Equation 2.44 the following can be shown:

\[ S_{11}^S S_{22}^S = \frac{S_{MS}^{11} S_{MS}^{22}}{S_{MT}^{21} S_{MT}^{12}} e^{-2jK_0d} \] (2.47)

Since \( S_{11}^S \) and \( S_{22}^S \) are theoretically equal, \( S_{11}^S \) can be isolated by taking the square root of Equation 2.47 \[8\].

\[ S_{11}^S = S_{22}^S = \sqrt{S_{11}^S S_{22}^S} = e^{-jK_0d} \sqrt{\frac{S_{MS}^{11}}{S_{MT}^{21}}} \left| \frac{S_{MS}^{22}}{S_{MT}^{12}} \right| \exp \left\{ \frac{j}{2} \left( \frac{S_{11}^S}{S_{MT}^{21}} + \frac{S_{12}^S}{S_{MT}^{12}} \right) \right\} \] (2.48)

A similar method can be followed for \( S_{21}^S \) and \( S_{12}^S \) to give an average of the two values, helping to remove any anomalies in the data.

\[ S_{21}^S = S_{12}^S = \sqrt{S_{21}^S S_{12}^S} = e^{-jK_0d} \sqrt{\frac{S_{MS}^{21}}{S_{MT}^{21}}} \left| \frac{S_{MS}^{12}}{S_{MT}^{12}} \right| \exp \left\{ \frac{j}{2} \left( \frac{S_{21}^S}{S_{MT}^{21}} + \frac{S_{12}^S}{S_{MT}^{12}} \right) \right\} \] (2.49)

### 2.11 Oblique Incident S Parameter Isolation

When working with a rotated sample, the thickness of the material is no longer able to be used when isolating the sample S parameters.
Figure 2.13 depicts a material of thickness $d$, with an angle of incidence $\theta$, where the EM wave travels from the front plane of the sample (A) to the back plane of the sample (B). The direct path through the sample is depicted as $h$ and the tangential path from the non-oblique incidence of $t$. Figure 2.13 shows an EM wave passing through a medium at an angle does not travel the distance $d$ through it and so requires a modified calibration technique [3] resulting in the following equations.

$$h = \frac{d}{\cos(\theta)} \quad t = d \cdot \tan(\theta)$$  \hspace{1cm} (2.50)

The propagation constant can then be separated into two individual parts, the $\hat{z}$ direction (normal to the surface) and the $\hat{p}$ direction (parallel to the surface).

$$\vec{K}_0 = \hat{p}K_{p0} + \hat{z}K_{z0} = \hat{p}K_0 \sin \theta + \hat{z}K_0 \cos \theta$$  \hspace{1cm} (2.51)

The measured $S$ parameter with and without the sample is then changed to the following:

$$S_{21}^{MS} = S_{21}^A S_{21}^S e^{-jK_{p0}t} S_{21}^B$$  \hspace{1cm} (2.52)
\[ S_{21}^S = \frac{S_{21}^{MS}}{S_{21}^{MT} e^{-j p_0 t}} \] (2.53)

\[ S_{21}^{MT} = S_{21}^A e^{-j K_0 h} S_{21}^B \] (2.54)

Using Equation 2.54 and Equation 2.53, the S parameters for the sample can be isolated

\[ S_{21}^S = \frac{S_{21}^{MS}}{S_{21}^{MT} e^{j K_0 d} e^{-j K_{0d} t}} \] (2.55)

and simplified to

\[ S_{21}^S = \frac{S_{21}^{MS}}{S_{21}^{MT}} e^{-j K_{0d} \cos \theta} \] (2.56)

### 2.12 Snell’s Law

When a wave enters a medium at an oblique angle that possess a different index of refraction than the medium it was previously traveling through, it will experience a change in its angle of propagation. The relationship between the index of refraction and the angle of propagation is described in Equation 2.57; where \( n_1 \) and \( \theta_1 \) are the index of refraction and angle of propagation through material one and \( n_2 \) and \( \theta_2 \) are the index of refraction and angle of propagation through material two.

\[ n_1 \sin \theta_1 = n_2 \sin \theta_2 \] (2.57)

Equation 2.57 is referred to as Snell’s Law and is depicted in Figure 2.14 [9].

The index of refraction is inversely related to the EM wave’s velocity through the material. With this, Equation 2.57 can be rearranged into equation 2.58 where \( V_1 \) is the velocity of light in material one and \( V_2 \) is the velocity of light in material two.

\[ \frac{\sin \theta_1}{V_1} = \frac{\sin \theta_2}{V_2} \] (2.58)

Equation 2.58 can then be rearranged to solve for \( \theta_2 \).

\[ \theta_2 = \arcsin \frac{V_2}{V_1} \sin \theta_1 \] (2.59)
The velocity of EM waves can be defined by Equation 2.60 for purely real $\mu, \epsilon$ [2].

$$V = \frac{1}{\sqrt{\mu \epsilon}}$$  \hspace{1cm} (2.60)

Combining Equation 2.60 and Equation 2.59 gives the following:

$$\theta_2 = \arcsin \left( \frac{\sqrt{\mu_1 \epsilon_1}}{\sqrt{\mu_2 \epsilon_2}} \sin \theta_1 \right)$$  \hspace{1cm} (2.61)

When medium one is assumed to be free space, Equation 2.61 simplifies to

$$\theta_2 = \arcsin \left( \frac{\sin \theta_1}{\sqrt{\mu_2 \epsilon_2}} \right)$$  \hspace{1cm} (2.62)

where $\epsilon_{2r}$ and $\mu_{2r}$ are the relative permittivity and permeability of material two.

If there is more than one material stacked together, the angle in the third material can be found by substituting Equation 2.62 into Equation 2.61 solving for $\theta_3$.

$$\theta_3 = \arcsin \left[ \frac{\sqrt{\mu_2 \epsilon_2}}{\sqrt{\mu_3 \epsilon_3}} \sin \left( \arcsin \left( \frac{\sin \theta_1}{\sqrt{\mu_2 \epsilon_2}} \right) \right) \right]$$  \hspace{1cm} (2.63)

were $\theta_3$ is the propagation angle in the second material. Equation 2.63 can then be simplified to

$$\theta_3 = \arcsin \left( \frac{\sin \theta_1}{\sqrt{\mu_3 \epsilon_3}} \right)$$  \hspace{1cm} (2.64)

Equation 2.64 shows that the angle of travel inside a material surrounded by free space is not dependent on other materials stacked before or after it and only the angle of travel in free space.
3. Mathematical Derivation

Building upon the theoretical background presented in chapter 2, this chapter will discuss the EM effects on a biaxial sample that is not aligned with the axes of the antenna and the different techniques needed to calibrate the resulting S parameters [10].

3.1 Biaxial Anisotropic Sample with Oblique Incidence Y Axis

This section will discuss a biaxial sample rotated around the Y axis where the angle of rotation is known (Figure 3.1).

3.1.1 Rotated Material Parameters Around Y axis

When a biaxial sample is rotated around the Y axis (Figure 3.1), the \( \hat{z} \) material properties enter the path of \( \hat{x} \) orientated EM fields. This new orientation changes the perceived material parameters to become a combination of both the \( \epsilon_{x'x'} \), \( \mu_{x'x'} \) and \( \epsilon_{z'z'} \), \( \mu_{z'z'} \). Since the Y axis is not affected by this rotation, it can be ignored when calculating the new perceived material parameters. \( \epsilon_{x'x'} \), \( \mu_{x'x'} \) and \( \epsilon_{z'z'} \), \( \mu_{z'z'} \) can be found using the following transformation:

\[
\begin{align*}
\mathbf{T}_y^{-1} & = \begin{pmatrix}
\hat{x}' \cdot \hat{x} & \hat{x}' \cdot \hat{z} \\
\hat{z}' \cdot \hat{x} & \hat{z}' \cdot \hat{z}
\end{pmatrix} \\
& = \begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}
\end{align*}
\]

(3.1)

with \( \hat{\epsilon}_\theta \) being the rotated permittivity around the Y axis \( \theta \) degrees and \( \mathbf{T} \) being the transformation tensor equal to

\[
\begin{align*}
\mathbf{T}_y & = \begin{pmatrix}
\hat{x}' \cdot \hat{x} & \hat{x}' \cdot \hat{z} \\
\hat{z}' \cdot \hat{x} & \hat{z}' \cdot \hat{z}
\end{pmatrix} \\
& = \begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}
\end{align*}
\]

(3.2)

and the inverse T
\[ \tilde{T}^{-1}_{y} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \] (3.3)

Using Equation 3.1 with Equation 3.2 and 3.3 gives the new perceived permittivity after the rotation.

\[ \tilde{\epsilon}_{\theta} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \epsilon_{xx} & 0 \\ 0 & \epsilon_{zz} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \] (3.4)

\[ \tilde{\epsilon}_{\theta} = \begin{pmatrix} \epsilon_{xx} \cos^2(\theta) + \epsilon_{zz} \sin^2(\theta) & \sin \theta \cos \theta (\epsilon_{xx} - \epsilon_{zz}) \\ \sin \theta \cos \theta (\epsilon_{xx} - \epsilon_{zz}) & \epsilon_{xx} \sin^2(\theta) + \epsilon_{zz} \cos^2(\theta) \end{pmatrix} \] (3.5)

The same method can be used to find the rotated permeability.

\[ \tilde{\mu}_{\theta} = \begin{pmatrix} \mu_{xx} \cos^2(\theta) + \mu_{zz} \sin^2(\theta) & \sin \theta \cos \theta (\mu_{xx} - \mu_{zz}) \\ \sin \theta \cos \theta (\mu_{xx} - \mu_{zz}) & \mu_{xx} \sin^2(\theta) + \mu_{zz} \cos^2(\theta) \end{pmatrix} \] (3.6)

As can be seen from Equation 3.5 and Equation 3.6, the non-diagonal material properties are no longer equal to zero. Since the non-diagonal material parameters greatly increase the complexity of the following calculations, it is simpler to assume that the antenna had been rotated instead of the sample by applying the transformation to the measured S parameters.

### 3.1.2 Solve Y Axis Rotated S Parameters

The S parameters follow the same transformation as the material parameters leading to the following:

\[ \tilde{S}^{S}_{\theta} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} S^{xxS} & 0 \\ 0 & S^{zzS} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \] (3.7)

\[ \tilde{S}^{S}_{\theta} = \begin{pmatrix} S^{xxS} \cos^2(\theta) + S^{zzS} \sin^2(\theta) & \sin \theta \cos \theta (S^{xxS} - S^{zzS}) \\ \sin \theta \cos \theta (S^{xxS} - S^{zzS}) & S^{xxS} \sin^2(\theta) + S^{zzS} \cos^2(\theta) \end{pmatrix} \] (3.8)

Using Equation 3.8, the S parameters correlating to the Z axis is introduced into the measurements without introducing non-diagonal material parameters into the calculations.
Figure 3.2 depicts a sample that has been rotated around the Y axis and the X axis lines up with the surface plane of the sample. A parallel E field ($\vec{E}^\parallel_i$) and H field ($\vec{H}^\parallel_i$), propagating in the $\hat{k}$ direction in free space ($\mu_0, \varepsilon_0$), hit the boundary of a sample at an oblique incidence of angle $\theta$. Part of the field then reflects off of the sample at an angle of $\theta$ from the initial incidence as E field ($\vec{E}^\parallel_T$) and H field ($\vec{H}^\parallel_T$). The rest of the field travels through the sample with material parameter $\mu, \varepsilon$ at an internal angle of $\theta_i$ and emerges through the opposite side of the sample at an angle of $\theta$, where the transmitted E field ($\vec{E}^\perp_T$) and transmitted H field ($\vec{H}^\perp_T$) continue propagating forward in free space ($\mu_0, \varepsilon_0$). The image also depicts the same interaction but for a perpendicular E field ($\vec{E}^\perp_i$) and H field ($\vec{E}^\perp_i$).

### 3.1.3 Field Analysis

A traveling EM wave with both parallel and perpendicular orientations traveling in the $\hat{k}$ direction (Figure 3.2) will have a propagation constant equal to the following [5] [11]:

$$
\vec{K} = \hat{x}K_x + \hat{z}K_z
$$

where $K_x$ and $K_z$ equal

$$
K_x = K \sin \theta \quad K_z = K \cos \theta
$$
The cross product of the propagation constant and the identity matrix is then equal to

\[
\vec{K} = \vec{K}X\vec{I} = \begin{pmatrix} K_x \\ 0 \\ K_z \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -K_z & 0 \\ K_z & 0 & -K_x \\ 0 & K_x & 0 \end{pmatrix}
\]

(3.11)

The E and H field are then assumed to equal

\[
\vec{E}(\vec{r}) = \vec{E}_0 e^{-j\hat{k}\vec{r}} \quad \vec{H}(\vec{r}) = \vec{H}_0 e^{-j\hat{k}\vec{r}}
\]

(3.12)

Equation 3.11 and 3.12 combined with Ampere’s Law (Equation 2.2) gives:

\[-j\vec{K} \cdot \vec{E} = -j\omega\vec{\mu} \cdot \vec{H} \]

(3.13)

Applying the same concept but with Faraday’s Law (Equation 2.1) gives:

\[-j\vec{K} \cdot \vec{H} = j\omega\vec{\varepsilon} \cdot \vec{E} \]

(3.14)

Solving Equation 3.13 for \(\vec{H}\)

\[\vec{H} = \frac{1}{\omega}\vec{\mu}^{-1} \cdot \vec{K} \cdot \vec{E} \]

(3.15)

and substituting this into equation 3.14, then rearranging terms gives

\[[\vec{K} \cdot \vec{\mu}^{-1} \cdot \vec{K} + \omega^2\vec{\varepsilon}] \cdot \vec{E} = 0 \]

(3.16)

The E field relationship can also be represented as

\[\vec{W}_E \cdot \vec{E} = 0 \]

(3.17)

Solving this equation gives

\[
\begin{pmatrix} -K_z^2 + \omega^2\varepsilon_{xx}\mu_{yy} & 0 & K_z K_x \\ 0 & -K_z^2\mu_{zz} - K_z^2\mu_{xx} + \omega^2\varepsilon_{yy}\mu_{xx}\mu_{zz} & 0 \\ K_z K_x & 0 & -K_x^2 + \omega^2\varepsilon_{zz}\mu_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0
\]

(3.18)
Next, column 2 and 3 are switched, then row 2 and 3 are switched.

\[
\begin{pmatrix}
-K_z^2 + \omega^2 \epsilon_{xx} \mu_{yy} & K_z K_x & 0 \\
K_z K_x & -K_z^2 + \omega^2 \epsilon_{zz} \mu_{yy} & 0 \\
0 & 0 & -K_z^2 \mu_{zz} - K_z^2 \mu_{xx} + \omega^2 \epsilon_{yy} \mu_{xx} \mu_{zz}
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_z \\
E_y
\end{pmatrix} = 0 \quad (3.19)
\]

Equation 3.19 can then be split into \(E_x, E_z\)

\[
\begin{pmatrix}
-K_z^2 + \omega^2 \epsilon_{xx} \mu_{yy} & K_z K_x \\
K_z K_x & -K_z^2 + \omega^2 \epsilon_{zz} \mu_{yy}
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_z
\end{pmatrix} = 0 \quad (3.20)
\]

and the \(E_y\) components.

\[
\begin{pmatrix}
-K_z^2 \mu_{zz} - K_z^2 \mu_{xx} + \omega^2 \epsilon_{yy} \mu_{xx} \mu_{zz}
\end{pmatrix}
\begin{pmatrix}
E_y
\end{pmatrix} = 0 \quad (3.21)
\]

Since setting \(E(\vec{r}) = E_0 e^{-j\vec{k}\vec{r}} = 0\) is a trivial solution, a non-trivial solution can be obtained if the determinant \(\hat{W} = 0\). When applied to Equation 3.20, the following is found.

\[
(-K_z^2 + \omega^2 \epsilon_{xx} \mu_{yy})(-K_z^2 + \omega^2 \epsilon_{zz} \mu_{yy}) - (K_z K_x)(K_z K_x) = 0 \quad (3.22)
\]

The propagating constant can then be solved giving:

\[
K^\parallel = \pm \sqrt{\frac{\omega^2 \epsilon_{xx} \epsilon_{zz} \mu_{yy}}{\cos^2(\theta) \epsilon_{zz} + \sin^2(\theta) \epsilon_{xx}}} \quad (3.23)
\]

Equation 3.23 represents the parallel propagation constant. Repeating for Equation 3.21 gives the perpendicular propagation constant.

\[
K^\perp = \pm \sqrt{\frac{\omega^2 \epsilon_{yy} \mu_{xx} \mu_{zz}}{\cos^2(\theta) \mu_{zz} + \sin^2(\theta) \mu_{xx}}} \quad (3.24)
\]

The \(\pm\) in the above equations represents the forward (+) and reverse (-) traveling waves. Equation 3.23 and Equation 3.24 represent the propagation constant that the EM wave experiences while traveling through a biaxial material rotated about the Y axis.
For non-oblique incidence ($\theta = 0$), the propagation constant simplifies to

\[ K_{\parallel} = \pm \sqrt{\omega^2 \epsilon_{xx} \mu_{yy}} \]  
\[ K_{\perp} = \pm \sqrt{\omega^2 \epsilon_{yy} \mu_{xx}} \]  

(3.25a)  
(3.25b)

and for free space, the equations further simplify to

\[ K_{\parallel 0} = \pm \sqrt{\omega^2 \epsilon_0 \mu_0} = \pm \frac{\omega}{C} \]  
\[ K_{\perp 0} = \pm \sqrt{\omega^2 \epsilon_0 \mu_0} = \pm \frac{\omega}{C} \]  

(3.26a)  
(3.26b)

In free space, both the parallel and perpendicular propagation constants are the same. This is what would be expected as isotropic/free space mediums do not depend on direction.

For simplicity, $K_z$ and $K_x$ will be used in the following calculations depicted in Equation 3.27.

\[ K_z^\parallel = \sqrt{\omega^2 \epsilon_{xx} \mu_{yy} - K_x^2 \frac{\epsilon_{xx}}{\epsilon_{zz}}} \]  
\[ K_z^\parallel = \sqrt{\omega^2 \epsilon_{gg} \mu_{gg} - K_x^2 \frac{\epsilon_{gg}}{\epsilon_{gg}}} \]  

(3.27)

3.1.4 Electric Field Relationship

When Equation 3.23 is incorporated into Equation 3.21, the resulting eigenvector results in a trivial solution, forcing $E_y = 0$ for the parallel case. This allows the relationship between $E_x$ and $E_z$ to be solved independently. When Equation 3.24 is incorporated into 3.20, a similar result is produced, making $E_x = 0$ and $E_z = 0$. This means that $E_y$ can also be solved independently.

The relationship between $E_x$ and $E_z$ for the parallel case can be found by taking Equation 3.20 and inserting Equation 3.27.

\[ \begin{pmatrix} K_x^2 \frac{\epsilon_{xx}}{\epsilon_{zz}} - \omega^2 \epsilon_{xx} \mu_{yy} + \omega^2 \epsilon_{yy} \mu_{yy} \\ \pm K_x K_z \\ \pm K_x K_z \\ \pm K_x \frac{\epsilon_{xx}}{\epsilon_{zz}} - \omega^2 \epsilon_{zz} \mu_{yy} + \omega^2 \epsilon_{yy} \mu_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_z \\ E_z \end{pmatrix} = 0 \]  

(3.28)

Next, multiply row one by $K_z$ and row two by $K_x \frac{\epsilon_{xx}}{\epsilon_{zz}}$

\[ \begin{pmatrix} \pm K_x^2 K_z \frac{\epsilon_{xx}}{\epsilon_{zz}} & K_x^3 K_z \\ \pm K_x^2 K_z \frac{\epsilon_{xx}}{\epsilon_{zz}} & K_x^3 K_z \\ \pm K_x^2 K_z \frac{\epsilon_{xx}}{\epsilon_{zz}} & K_x^3 K_z \end{pmatrix} \begin{pmatrix} E_x \\ E_z \end{pmatrix} = 0 \]  

(3.29)
This shows that both eigenvectors of $E_x$ and $E_z$ in Equation 3.20 are just transformations of the same relationship.

Using Equation 3.29, the relationship between $E_x$ and $E_z$ can be found.

$$\pm K_x^2 K_z \frac{\epsilon_{xx}}{\epsilon_{zz}} E_x + K z^2 K_x E_z = 0$$  \hspace{1cm} (3.30)

$$E_z = \mp \frac{K_x \epsilon_{xx}}{K_z \epsilon_{zz}} E_x$$  \hspace{1cm} (3.31)

The $E$ field inside the biaxial sample for the parallel case can then be shown to be equal to

$$\vec{E}^\parallel = (\hat{x} \mp \hat{z} \frac{K_x \epsilon_{xx}}{K_z \epsilon_{zz}}) E_x e^{\mp j K^\parallel k}$$  \hspace{1cm} (3.32)

For the perpendicular case, when Equation 3.24 is incorporated into the eigenvector of Equation 3.21, it equals zero. This forces the $\vec{E}^\perp$ to be

$$\vec{E}^\perp = \hat{y} E_y e^{\mp j K^\perp k}$$  \hspace{1cm} (3.33)

### 3.1.5 Magnetic Field Relationship

Once the $E$ field relationship has been calculated, the $H$ field can be found using Equation 3.15. For the parallel case, this results in the following:

$$\begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \frac{1}{\omega} \begin{pmatrix} \frac{1}{\mu_{xx}} & 0 & 0 \\ 0 & \frac{1}{\mu_{yy}} & 0 \\ 0 & 0 & \frac{1}{\mu_{zz}} \end{pmatrix} \begin{pmatrix} 0 & \mp K_x & 0 \\ \pm K_z & 0 & \mp K_x \\ 0 & \pm K_x & 0 \end{pmatrix} \begin{pmatrix} E_x \\ 0 \\ \mp \frac{K_x \epsilon_{xx}}{K_z \epsilon_{zz}} E_x \end{pmatrix}$$  \hspace{1cm} (3.34)

$$\begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} 0 \\ \pm \frac{E_x K_{yy}}{\mu_{yy} \omega} \pm \frac{K_x \epsilon_{xx} E_x K_z}{\mu_{yy} \omega} \\ 0 \end{pmatrix}$$  \hspace{1cm} (3.35)

Equation 3.35 shows that $\vec{H}$ for the parallel case only has a $y$ directed component with the following relationship.

$$\vec{H}^\parallel = \pm \hat{y} \left( \frac{\omega \epsilon_{yy}}{K_z} \right) E_x e^{\mp j K^\parallel k}$$  \hspace{1cm} (3.36)
For the perpendicular case, $\vec{H}$ is equal to

$$
\begin{pmatrix}
H_x \\
H_y \\
H_z
\end{pmatrix} = \frac{1}{\omega}
\begin{pmatrix}
\frac{1}{\mu_{xx}} & 0 & 0 \\
0 & \frac{1}{\mu_{yy}} & 0 \\
0 & 0 & \frac{1}{\mu_{zz}}
\end{pmatrix}
\begin{pmatrix}
0 & \mp K_z & 0 \\
\pm K_z & 0 & \mp K_x \\
0 & \mp K_x & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
E_y \\
0
\end{pmatrix} \quad (3.37)
$$

Equation 3.38 shows that for the perpendicular case, the $\vec{H}$ exists in both the $X$ and $Z$ axis. $\vec{H}^\perp$ can then be shown to equal

$$
\vec{H}^\perp = \mp \frac{K_z}{\mu_{xx}\omega} (\hat{x} \pm \frac{K_z}{K^\perp_z \mu_{zz}} \hat{z}) E_y e^{\mp j K^\perp k} \quad (3.39)
$$

### 3.1.6 Impedance

The wave impedance of a medium is defined as

$$
Z_w = \frac{E_{\text{lang}}}{H_{\text{lang}}} \quad (3.40)
$$

This equation relates the ratio between $\vec{E}$ and $\vec{H}$ traveling along the same direction. To find the impedance along the $X$-$Z$ axis, $\vec{E}$ and $\vec{H}$ traveling in the $X$-$Z$ axis will need to be used. Taking Equation 3.39 and Equation 3.32 gives the perpendicular impedance as it relates to the perpendicular magnetic field.

$$
Z_w^\perp = \frac{\mu_{xx}\omega}{K^\perp_z} = \frac{\mu_{xx}\omega}{K^\perp \cos(\theta)} = \eta^\perp \quad (3.41)
$$

For the parallel impedance, $\vec{E}$ and $\vec{H}$ in the $Y$ axis will be used. Taking Equation 3.36 and Equation 3.33 the parallel impedance is equal to

$$
Z_w^\parallel = \frac{K^\parallel_z}{\omega \varepsilon_{yy}} = \frac{K^\parallel \cos(\theta)}{\omega \varepsilon_{yy}} = \eta^\parallel \quad (3.42)
$$

For free space impedance, the equations can be simplified to the following:
\[ Z_{w0} = \frac{K_0}{\omega \varepsilon_0} = \frac{\omega \sqrt{\varepsilon_0 \mu_0}}{\omega \varepsilon_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0 \] (3.43)

This equation remains the same for both perpendicular and parallel cases.

### 3.1.7 Parallel Polarization: Reflection and Transmission

The layout of the fields passing through a biaxial material are defined in Figure 3.2. The fields are divided into incident \((E^i, H^i)\), reflected \((E^r, H^r)\) and transmitted \((E^t, H^t)\) portions within each region. The total reflection is defined as \(\Gamma (S_{11})\) and the total transmission is defined as \(T (S_{21})\). The internal reflection is defined as \(r\) and the internal transmission is defined as \(t\).

Region 1 Fields (Free-Space):

\[
\vec{E}^i = \left(\hat{x} - \hat{z} \frac{K_x}{K_z}\right) E_x e^{-jK_k} \tag{3.44}
\]
\[
\vec{H}^i = \frac{E_x}{\eta_0} e^{-jK_k} \tag{3.45}
\]
\[
\vec{E}^{\Gamma} = \left(\hat{x} + \hat{z} \frac{K_x}{K_z}\right) \Gamma E_x e^{jK_k} \tag{3.46}
\]
\[
\vec{H}^{\Gamma} = -\frac{E_x}{\eta_0} \Gamma e^{jK_k} \tag{3.47}
\]

Region 2 Fields (Biaxial Anisotropic Slab):

\[
\vec{E}^i = \left(\hat{x} - \hat{z} \frac{\epsilon_{xx} K_x}{\epsilon_{zz} K_z}\right) t^i E_x e^{-jK_k} \tag{3.48}
\]
\[
\vec{H}^i = \frac{E_x}{\eta_i} e^{-jK_k} \tag{3.49}
\]
\[
\vec{E}^{r} = \left(\hat{x} + \hat{z} \frac{\epsilon_{xx} K_x}{\epsilon_{zz} K_z}\right) r^i E_x e^{jK_k} \tag{3.50}
\]
\[
\vec{H}^{r} = -\frac{E_x}{\eta_i} e^{jK_k} \tag{3.51}
\]
Region 3 Fields (Free-Space):

\[
\vec{E}^T = \left( \hat{x} - \frac{x K}{K^2} \right) T^x e^{-jK^1(k - d\cos(\theta))} 
\]  (3.52)

\[
\vec{H}^T = \eta_0 T^x e^{-jK^1(k - d\cos(\theta))} 
\]  (3.53)

Enforcing the continuity of tangential \( \vec{E} \) and \( \vec{H} \) across interface 1 (\( k=0 \)), the point that the EM waves enter the biaxial sample, and interface 2 (\( k = d\cos(\theta) \)), the point where the same EM waves leave the biaxial material, gives the following relationships.

\[
E^\parallel(0^-) = E^\parallel(0^+) \quad H^\parallel(0^-) = H^\parallel(0^+) 
\]  (3.54a)

\[
E^\parallel(d\cos(\theta)^-) = E^\parallel(d\cos(\theta)^+) \quad H^\parallel(d\cos(\theta)^-) = H^\parallel(d\cos(\theta)^+) 
\]  (3.54b)

Interface 1 (\( k=0 \))

\[
1 + \Gamma^\parallel = t^\parallel + r^\parallel 
\]  (3.55)

\[
\frac{1}{\eta_0} (1 - \Gamma^\parallel) = \frac{1}{\eta^\parallel} (t^\parallel - r^\parallel) 
\]  (3.56)

Interface 2 (\( k = d\cos(\theta) \))

\[
t^\parallel e^{-jK^1d\cos(\theta)} + r^\parallel e^{jK^1d\cos(\theta)} = T^\parallel 
\]  (3.57)

\[
\frac{\eta_0}{\eta^\parallel} \left( t^\parallel e^{-jK^1d\cos(\theta)} - r^\parallel e^{jK^1d\cos(\theta)} \right) = T^\parallel 
\]  (3.58)

Equation 3.57 and Equation 3.58 can be equated to give

\[
\frac{\eta_0}{\eta^\parallel} \left( t^\parallel e^{-jK^1d\cos(\theta)} - r^\parallel e^{jK^1d\cos(\theta)} \right) = t^\parallel e^{-jK^1d\cos(\theta)} + r^\parallel e^{jK^1d\cos(\theta)} 
\]  (3.59)

Next, the one-way phase delay and attenuation experienced by the wave as it travels through the thickness of the sample for parallel polarization (\( P^\parallel \)) will be defined as
\[ P^\parallel = e^{-jK^1 \cos(\theta)} \]  

(3.60)

substituting Equation 3.60 into Equation 3.59 gives:

\[ \frac{\eta_0}{\eta} \left( t^\parallel P^\parallel - r^\parallel P^{-1^\parallel} \right) = t^\parallel P^\parallel + r^\parallel P^{-1^\parallel} \]  

(3.61)

Next, multiply Equation 3.61 by \( P^\parallel \)

\[ \eta_0 t^\parallel P^2^\parallel - \eta_0 r^\parallel = \eta^\parallel t^\parallel P^2^\parallel + \eta^\parallel r^\parallel \]  

(3.62)

and then separate the \( t \) and \( r \) terms.

\[ - t^\parallel P^2^\parallel (\eta^\parallel - \eta_0) = r^\parallel (\eta_0 + \eta^\parallel) \]  

(3.63)

The internal reflection coefficient \( (R^\parallel) \) is then defined as

\[ R^\parallel = \frac{\eta^\parallel - \eta_0}{\eta^\parallel + \eta_0} \]  

(3.64)

which can be further simplified by dividing the equation by \( \mu_0 \), resulting in the following relationship where \( \eta_r \) is the relative impedance.

\[ R_r^\parallel = \frac{\eta_r^\parallel - 1}{\eta_r^\parallel + 1} \]  

(3.65)

Solving Equation 3.63 for \( r^\parallel \) using Equation 3.64 gives:

\[ r^\parallel = -t^\parallel P^2^\parallel R^\parallel \]  

(3.66)

Next, Equation 3.56 is divided by Equation 3.55 while inserting Equation 3.66 for \( r^\parallel \).

\[ \frac{\eta^\parallel (1 - \Gamma^\parallel)}{\eta_0 (1 + \Gamma^\parallel)} = \frac{1 + P^2^\parallel R}{1 - P^2^\parallel R} \]  

(3.67)

Multiply out Equation 3.67,

\[ \Gamma^\parallel P^2^\parallel R^\parallel (\eta^\parallel - \eta_0) - \Gamma^\parallel (\eta^\parallel + \eta_0) = P^2^\parallel R^\parallel (\eta^\parallel + \eta_0) - (\eta^\parallel - \eta_0) \]  

(3.68)
divide by $\eta_\parallel + \eta_0$ and solve for $\Gamma_\parallel$.

$$\Gamma_\parallel = \frac{R_\parallel(1 - P_\parallel^2)}{1 - P_\parallel^2 R_\parallel^2} \quad (3.69)$$

This gives the total reflection resulting from the biaxial material.

Next take Equation 3.55 and substitute it into Equation 3.66 and Equation 3.69.

$$1 + \frac{R_\parallel(1 - P_\parallel^2)}{1 - P_\parallel^2 R_\parallel^2} = t(1 - P_\parallel^2 R_\parallel^2) \quad (3.70)$$

Solving for $t$

$$t = \frac{(1 - P_\parallel^2 R_\parallel^2)(1 + R_\parallel)}{(1 - P_\parallel^2 R_\parallel^2)(1 - P_\parallel^2 R_\parallel^2)} \quad (3.71)$$

and simplifying gives the following:

$$t = \frac{1 + R_\parallel}{1 - P_\parallel^2 R_\parallel^2} \quad (3.72)$$

Lastly, take Equation 3.57 and combine Equation 3.60, Equation 3.66 and Equation 3.72 gives the following relationship for the total transmission experienced by the biaxial sample.

$$T_\parallel = \frac{P_\parallel(1 - R_\parallel^2)}{1 - P_\parallel^2 R_\parallel^2} \quad (3.73)$$

3.1.8 Perpendicular Polarization: Reflection and Transmission

Repeating the same process for the perpendicular case:

Region 1 Fields (Free-Space):

$$\vec{E}^{i\perp} = \hat{y} E_y e^{-jK^{\perp}k} \quad (3.74)$$

$$\vec{H}^{i\perp} = -\left(\hat{x} - \hat{z} \frac{K^{\perp}}{\eta_0} \right) \frac{E_y e^{-jK^{\perp}k}}{\eta_0} \quad (3.75)$$

$$\vec{E}^{i\perp} = \hat{y} \Gamma^{\perp} E_y e^{jK^{\perp}k} \quad (3.76)$$
\[ \bar{H}^{\perp} = \left( \hat{x} + \frac{\mu_{x z}}{\mu_{z z}} \frac{K_x}{K_z} \right) \Gamma^{\perp} \frac{E_y}{\eta_0} e^{j K^+ k} \]  

Region 2 Fields (Biaxial Anisotropic Slab):

\[ \vec{E}^{\perp} = \hat{y} t^{\perp} E_y e^{-j K^+ k} \]  

\[ \vec{H}^{\perp} = -\left( \hat{x} - \frac{\hat{z} \mu_{x z} K_x}{\mu_{z z} K_z} \right) t^{\perp} \frac{E_y}{\eta^{\perp}} e^{-j K^+ k} \]  

\[ \vec{E}^{r^{\perp}} = \hat{y} r^{\perp} E_y e^{j K^+ k} \]  

\[ \vec{H}^{r^{\perp}} = \left( \hat{x} + \frac{\hat{z} \mu_{x z} K_x}{\mu_{z z} K_z} \right) r^{\perp} \frac{E_y}{\eta^{\perp}} e^{j K^+ k} \]  

Region 3 Fields (Free-Space):

\[ \vec{E}^{T^{\perp}} = \hat{y} T^{\perp} E_y e^{-j K^+(k - d \cos(\theta))} \]  

\[ \vec{H}^{T^{\perp}} = -\left( \hat{x} - \frac{\hat{z} K_x}{K_z^+} \right) T^{\perp} \frac{E_y}{\eta_0} e^{-j K^+(k - d \cos(\theta))} \]  

Enforcing the continuity of tangential \( \vec{E} \) and \( \vec{H} \) across interface 1 (\( k=0 \)) and interface 2 (\( k = d \cos(\theta) \)) the previous equations can be reduced using the following boundary conditions:

\[ E^{\perp}(0^-) = E^{\perp}(0^+) \quad H^{\perp}(0^-) = H^{\perp}(0^+) \]  

\[ E^{\perp}(d \cos(\theta)^-) = E^{\perp}(d \cos(\theta)^+) \quad H^{\perp}(d \cos(\theta)^-) = H^{\perp}(d \cos(\theta)^+) \]  

Interface 1 (\( k=0 \))

\[ 1 + \Gamma^{\perp} = t^{\perp} + r^{\perp} \]  

\[ \frac{1}{\eta_0} (1 - \Gamma^{\perp}) = \frac{1}{\eta^{\perp}} (t^{\perp} - r^{\perp}) \]
Interface 2 (k=dcos(θ))

\[ t^{\perp} e^{-jK^{\perp}d\cos(\theta)} + r^{\perp} e^{jK^{\perp}d\cos(\theta)} = T^{\perp} \]  \hspace{1cm} (3.87)

\[ \frac{\eta_0}{\eta^{\perp}} \left( t^{\perp} e^{-jK^{\perp}d\cos(\theta)} - r^{\perp} e^{jK^{\perp}d\cos(\theta)} \right) = T^{\perp} \]  \hspace{1cm} (3.88)

Since the above equations match the parallel version, there is no difference between the calculations except for having

\[ P^{\perp} = e^{-jK^{\perp}d\cos(\theta)} \]  \hspace{1cm} (3.89)

and

\[ R^{\perp} = \frac{\eta^{\perp} - \eta_0}{\eta^{\perp} + \eta_0} \hspace{0.5cm} R_{\perp} = \frac{\eta^{\perp} - 1}{\eta^{\perp} + 1} \]  \hspace{1cm} (3.90)

This gives the total reflection coefficient to be equal to

\[ \Gamma^{\perp} = \frac{R^{\perp}(1 - P^{2\perp})}{1 - P^{2\perp} R^{2\perp}} \]  \hspace{1cm} (3.91)

And transmission coefficient

\[ T^{\perp} = \frac{P^{\perp}(1 - R^{2\perp})}{1 - P^{2\perp} R^{2\perp}} \]  \hspace{1cm} (3.92)

Comparing Equations 3.92 and 3.91 with Equations 3.73 and 3.72 shows that the transmission and reflection relationships remain the same for both the perpendicular and parallel orientation for a biaxial sample rotated around the Y axis. The same process can be done for rotation around the X axis to get similar results.

### 3.2 Biaxial Sample with Oblique Incidence Z Axis

This section will discuss a biaxial sample rotated around the Z axis where the angle of rotation is unknown as shown in Figure 3.3.
3.2.1 Rotated Material Parameters Around Z Axis

When a biaxial sample is rotated so that the $\epsilon_{xx}, \epsilon_{yy}, \mu_{xx}, \mu_{yy}$ no longer align with the vertical and horizontal axes of the antenna, the $xx$ and $yy$ material parameters both affect the incoming $\hat{x}$ directed EM wave and the $\hat{y}$ directed EM wave as shown in Figure 3.3. The rotation creates a similar effect as a rotation around the Y axis shown in Equation 3.5 and Equation 3.6. This means that a rotation in the antenna will be used in the following calculations.

Figure 3.3. Rotated Axis Around Z Axis

3.2.2 Solve Rotated S Parameters Around Z Axis

To transform the S parameters for a sample ($S_S$), the following equation is used.

$$S_S' = T_z S'_S T_z^{-1}$$

(3.93)

with $T$ being the transformation tensor equal to

$$T_z = \begin{pmatrix} \hat{x}' \cdot \hat{x} & \hat{x}' \cdot \hat{y} \\ \hat{y}' \cdot \hat{x} & \hat{y}' \cdot \hat{y} \end{pmatrix} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

(3.94)

and the inverse $T$ equal to

$$T_z^{-1} = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

(3.95)

Using Equation 3.93 with Equation 3.94 and 3.95

$$S_S' = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} S_{xx} & 0 \\ 0 & S_{yy} \end{pmatrix} \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

(3.96)

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be isolated, changing the relationships discussed in Section 2.10 to the following:

\[
S_{\phi}^{x'x'S} = \left( \begin{array}{cc} S_{\phi}^{x'x'S} & S_{\phi}^{y'y'S} \\ S_{\phi}^{y'y'S} & S_{\phi}^{y'y'S} \end{array} \right) = \left( \begin{array}{cc} S^{xxS} \cos^2(\phi) + S^{yyS} \sin^2(\phi) & \sin \phi \cos \phi (S^{yyS} - S^{xxS}) \\ \sin \phi \cos \phi (S^{yyS} - S^{xxS}) & S^{xxS} \sin^2(\phi) + S^{yyS} \cos^2(\phi) \end{array} \right)
\]

(3.97)

To change the collected \( S' \) parameters shown in Equation 3.97 into their unrotated equivalency, the transformation done to the antenna will need to be reversed using the following equation.

\[
\hat{S} = \hat{T}^{-1} S_{\phi}^s \hat{T}
\]

(3.98)

\[
S_{\phi}^s = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} S_{\phi}^{x'x'S} \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}
\]

(3.99)

\[
S^{xxS} = \cos^2(\phi) S_{\phi}^{x'x'S} - \sin \phi \cos \phi (S_{\phi}^{y'y'S} + S_{\phi}^{x'x'S}) + \sin^2(\phi) S_{\phi}^{y'y'S}
\]

(3.100a)

\[
S^{yyS} = \cos \phi \sin \phi (S_{\phi}^{x'x'S} - S_{\phi}^{y'y'S}) + \cos^2(\phi) S_{\phi}^{x'x'S} - \sin^2(\phi) S_{\phi}^{y'y'S}
\]

(3.100b)

\[
S^{yxS} = \cos \phi \sin \phi (S_{\phi}^{x'x'S} - S_{\phi}^{y'y'S}) + \cos^2(\phi) S_{\phi}^{y'y'S} - \sin^2(\phi) S_{\phi}^{x'x'S}
\]

(3.100c)

\[
S^{yyS} = \sin^2(\phi) S_{\phi}^{x'x'S} + \sin \phi \cos \phi (S_{\phi}^{y'y'S} + S_{\phi}^{x'x'S}) + \cos^2(\phi) S_{\phi}^{y'y'S}
\]

(3.100d)

Equation 3.100 shows how to calculate the non-rotated \( S \) parameters from a biaxial sample rotated around the \( Z \) axis.

### 3.2.3 Calibrate Cross Polarized \( S \) parameters (Equivalency Method)

Equation 3.100 only remains true when the \( S \) parameters relate only to a biaxial sample. This means that the free space \( S \) parameters need to be removed from the measured data before they can be utilized. To do this, the \( S_{\phi}^{x'y'S}, S_{\phi}^{y'x'S}, S_{\phi}^{x'y'S}, S_{\phi}^{y'y'S} \) values for the biaxial sample need to be isolated, changing the relationships discussed in Section 2.10 to the following:

\[
S_{21}^{xxMS} = S_{21}^{xA} S_{21}^{x'x'S} S_{21}^{xxB} \quad S_{21}^{yyMS} = S_{21}^{yA} S_{21}^{y'y'S} S_{21}^{yyB}
\]

(3.101a)

\[
S_{21}^{yxMS} = S_{21}^{xA} S_{21}^{y'y'S} S_{21}^{yxB} \quad S_{21}^{yyMS} = S_{21}^{xA} S_{21}^{y'y'S} S_{21}^{yyB}
\]

(3.101b)
Where \( s^{MS} \) is the \( s \) parameter representing the total measurement with a sample in the FBS (Section A, Sample, and Section B). The total through measurements are then equal the following:

\[
\begin{align*}
S_{12}^{xxMS} &= S_{12}^{xxB} S_{12}^{x'S} S_{12}^{xxA} \quad & S_{12}^{yyMS} &= S_{12}^{yyB} S_{12}^{y'S} S_{12}^{yyA} \\
S_{12}^{xxMS} &= S_{12}^{yyB} S_{12}^{y'S} S_{12}^{xxA} \quad & S_{12}^{yyMS} &= S_{12}^{xxB} S_{12}^{x'S} S_{12}^{yyA}
\end{align*}
\] (3.102a)

\[
\begin{align*}
S_{11}^{xxMS} &= S_{11}^{xxA} S_{11}^{x'S} S_{11}^{xxA} \quad & S_{11}^{yyMS} &= S_{11}^{yyA} S_{11}^{y'S} S_{11}^{yyA} \\
S_{11}^{xxMS} &= S_{11}^{yyA} S_{11}^{y'S} S_{11}^{xxA} \quad & S_{11}^{yyMS} &= S_{11}^{xxA} S_{11}^{x'S} S_{11}^{yyA}
\end{align*}
\] (3.103a)

\[
\begin{align*}
S_{22}^{xxMS} &= S_{22}^{xxB} S_{22}^{x'S} S_{22}^{xxB} \quad & S_{22}^{yyMS} &= S_{22}^{yyB} S_{22}^{y'S} S_{22}^{yyB} \\
S_{22}^{xxMS} &= S_{22}^{yyB} S_{22}^{y'S} S_{22}^{xxB} \quad & S_{22}^{yyMS} &= S_{22}^{xxB} S_{22}^{x'S} S_{22}^{yyB}
\end{align*}
\] (3.104a)

\[
\begin{align*}
S_{21}^{xxMT} &= S_{21}^{xxA} e^{-j \kappa d} S_{21}^{xxB} \\
S_{21}^{yyMT} &= S_{21}^{yyA} e^{-j \kappa d} S_{21}^{yyB} \\
S_{12}^{xxMT} &= S_{12}^{xxB} e^{-j \kappa d} S_{12}^{xxA} \\
S_{12}^{yyMT} &= S_{12}^{yyB} e^{-j \kappa d} S_{12}^{yyA}
\end{align*}
\] (3.105a)

\[
\begin{align*}
S_{21}^{xxMS} &= S_{21}^{xxB} S_{21}^{x'S} S_{21}^{xxA} \quad & S_{21}^{yyMS} &= S_{21}^{yyB} S_{21}^{y'S} S_{21}^{yyA} \\
S_{21}^{xxMS} &= S_{21}^{yyB} S_{21}^{y'S} S_{21}^{xxA} \quad & S_{21}^{yyMS} &= S_{21}^{xxB} S_{21}^{x'S} S_{21}^{yyA}
\end{align*}
\] (3.106a)

\[
\begin{align*}
S_{21}^{xxMS} &= S_{21}^{xxB} S_{21}^{x'S} S_{21}^{xxA} \quad & S_{21}^{yyMS} &= S_{21}^{yyB} S_{21}^{y'S} S_{21}^{yyA} \\
S_{21}^{xxMS} &= S_{21}^{yyB} S_{21}^{y'S} S_{21}^{xxA} \quad & S_{21}^{yyMS} &= S_{21}^{xxB} S_{21}^{x'S} S_{21}^{yyA}
\end{align*}
\] (3.106b)

\[
\begin{align*}
S_{21}^{xxMT} &= S_{21}^{xxA} e^{-j \kappa d} S_{21}^{xxB} \\
S_{21}^{yyMT} &= S_{21}^{yyA} e^{-j \kappa d} S_{21}^{yyB} \\
S_{12}^{xxMT} &= S_{12}^{xxB} e^{-j \kappa d} S_{12}^{xxA} \\
S_{12}^{yyMT} &= S_{12}^{yyB} e^{-j \kappa d} S_{12}^{yyA}
\end{align*}
\] (3.107a)

\[
\begin{align*}
S_{21}^{xxMS} &= S_{21}^{xxB} S_{21}^{x'S} S_{21}^{xxA} \quad & S_{21}^{yyMS} &= S_{21}^{yyB} S_{21}^{y'S} S_{21}^{yyA} \\
S_{21}^{xxMS} &= S_{21}^{yyB} S_{21}^{y'S} S_{21}^{xxA} \quad & S_{21}^{yyMS} &= S_{21}^{xxB} S_{21}^{x'S} S_{21}^{yyA}
\end{align*}
\] (3.107b)

\[
\begin{align*}
S_{21}^{xxMS} &= S_{21}^{xxB} S_{21}^{x'S} S_{21}^{xxA} \quad & S_{21}^{yyMS} &= S_{21}^{yyB} S_{21}^{y'S} S_{21}^{yyA} \\
S_{21}^{xxMS} &= S_{21}^{yyB} S_{21}^{y'S} S_{21}^{xxA} \quad & S_{21}^{yyMS} &= S_{21}^{xxB} S_{21}^{x'S} S_{21}^{yyA}
\end{align*}
\] (3.107c)

\[
\begin{align*}
S_{21}^{xxMS} &= S_{21}^{xxB} S_{21}^{x'S} S_{21}^{xxA} \quad & S_{21}^{yyMS} &= S_{21}^{yyB} S_{21}^{y'S} S_{21}^{yyA} \\
S_{21}^{xxMS} &= S_{21}^{yyB} S_{21}^{y'S} S_{21}^{xxA} \quad & S_{21}^{yyMS} &= S_{21}^{xxB} S_{21}^{x'S} S_{21}^{yyA}
\end{align*}
\] (3.107d)
Since $S'_{11\phi}$, $S'_{11\phi}$, $S'_{21\phi}$, $S'_{11\phi}$ have the same equations as found in Section 2.10, the solutions found previously can be used as shown below.

\[
S'_{11\phi} = \sqrt{S_{11} S_{22}} = e^{-j K_0 d} \sqrt{S_{xxMS} S_{yyMS}} \exp \left\{ -j \left( S_{xxMS} + S_{yyMS} \right) \right\} \quad (3.110)
\]

\[
S'_{11\phi} = \sqrt{S_{11} S_{22}} = e^{-j K_0 d} \sqrt{S_{xxMS} S_{yyMS}} \exp \left\{ -j \left( S_{xxMS} + S_{yyMS} \right) \right\} \quad (3.111)
\]

\[
S'_{21\phi} = \sqrt{S_{21} S_{12}} = e^{-j K_0 d} \sqrt{S_{xxMS} S_{yyMS}} \exp \left\{ -j \left( S_{xxMS} + S_{yyMS} \right) \right\} \quad (3.112)
\]
For the reflection cross polarization terms, combine Equation 3.108b and Equation 3.109b, and then insert the results into Equation 3.105.

\[ S_{21\phi}^{y'y'} = \sqrt{S_{12}^{yyS} S_{21}^{yyS}} = e^{-jK_0d} \sqrt{\left| \frac{S_{21}^{yyMS}}{S_{21}^{yyMT}} \right| \left| \frac{S_{12}^{yyMS}}{S_{12}^{yyMT}} \right|} \exp\left\{ \frac{j}{2} \left( \angle S_{21}^{yyMS} + \angle S_{12}^{yyMS} \right) \right\} \]  
(3.113)

Solving for \( S_{11\phi}^{x'y'} \) and \( S_{22\phi}^{x'y'} \) gives the following.

\[ S_{11\phi}^{x'y'} = \sqrt{S_{11\phi}^{xxMS} S_{22\phi}^{xxMS}} = e^{-jK_0d} \sqrt{\left| \frac{S_{11}^{xxMS}}{S_{21}^{xxMT}} \right| \left| \frac{S_{12}^{xxMS}}{S_{12}^{xxMT}} \right|} \exp\left\{ \frac{j}{2} \left( \angle S_{11}^{xxMS} + \angle S_{12}^{xxMS} \right) \right\} \]  
(3.114)

\[ S_{22\phi}^{y'x'} = \sqrt{S_{11\phi}^{yxMS} S_{22\phi}^{yxMS}} = e^{-jK_0d} \sqrt{\left| \frac{S_{11}^{yxMS}}{S_{21}^{yxMT}} \right| \left| \frac{S_{12}^{yxMS}}{S_{12}^{yxMT}} \right|} \exp\left\{ \frac{j}{2} \left( \angle S_{11}^{yxMS} + \angle S_{12}^{yxMS} \right) \right\} \]  
(3.115)

For the transmission cross polarization terms, combining Equation 3.100b and Equation 3.100c gives:

\[ S_{11\phi}^{x'y'} = \sqrt{S_{11\phi}^{xxMS} S_{22\phi}^{xxMS}} = e^{-jK_0d} \sqrt{\left| \frac{S_{11}^{xxMS}}{S_{21}^{xxMT}} \right| \left| \frac{S_{12}^{xxMS}}{S_{12}^{xxMT}} \right|} \exp\left\{ \frac{j}{2} \left( \angle S_{11}^{xxMS} + \angle S_{12}^{xxMS} \right) \right\} \]  
(3.116)

\[ S_{21\phi}^{y'x'} = \sqrt{S_{11\phi}^{yxMS} S_{22\phi}^{yxMS}} = e^{-jK_0d} \sqrt{\left| \frac{S_{11}^{yxMS}}{S_{21}^{yxMT}} \right| \left| \frac{S_{12}^{yxMS}}{S_{12}^{yxMT}} \right|} \exp\left\{ \frac{j}{2} \left( \angle S_{11}^{yxMS} + \angle S_{12}^{yxMS} \right) \right\} \]  
(3.117)

Since Section A and Section B in Figure 2.12 are composed of air, which is an isotropic medium, the \( \hat{x} \) and \( \hat{y} \) are equivalent, allowing \( S_{xx} = S_{yy} \). The following substitutions can then be made in Equation 3.118 and simplified using Equation 3.105.
\[
\begin{align*}
S_{21}' & = S_{21} \exp \left\{ j \frac{K_0 d}{2} \left( \frac{S_{yyMS}}{S_{yymT}} + \frac{S_{yyMS}}{S_{yyMT}} \right) \right\} \\
S_{21}' & = S_{21} \exp \left\{ j \frac{K_0 d}{2} \left( \frac{S_{yyMS}}{S_{yyMT}} + \frac{S_{yyMS}}{S_{yyMT}} \right) \right\}
\end{align*}
\]

(3.120)

(3.121)

3.2.4 Calibrate Cross Polarized S Parameters (2 Samples)

Another method of calibrating the X-Y cross polarized \( S_{21} \) parameters is by using a second biaxial sample with known axes orientations. In the following calculations, the thickness of the second sample is assumed to be the same as the unknown sample. Using this, the vertical and horizontal S parameters for the second non-rotated biaxial sample are collected.

\[
\begin{align*}
S_{xxMS} & = S_{xxA} S_{xxB} \quad S_{xxMS} = S_{xxA} S_{xxB} e^{-j K_0 d} \\
S_{yyMS} & = S_{yyA} S_{yyB} \quad S_{yyMS} = S_{yyA} S_{yyB} e^{-j K_0 d}
\end{align*}
\]

(3.122a)

(3.122b)

Rearranging Equation 3.122 to solve for the biaxial sample and inserting into Equation 3.105 gives:

\[
\begin{align*}
S_{xxS2} & = S_{xxMS} S_{xxA} S_{xxB} \quad S_{xxS2} = S_{xxMS} S_{xxA} S_{xxB} e^{-j K_0 d} \\
S_{yyS2} & = S_{yyMS} S_{yyA} S_{yyB} \quad S_{yyS2} = S_{yyMS} S_{yyA} S_{yyB} e^{-j K_0 d}
\end{align*}
\]

(3.123a)

(3.123b)
Next, using Equation 3.123, the $S_{21\phi_2}^{x'y'}S_2^y$ and $S_{21\phi_2}^{x'y'}S_2^x$ can be calculated with Equation 3.97 for a Z axis rotation of $\phi_2$.

\[
S_{21\phi_2}^{x'y'}S_2^y = S_{21\phi_2}^{y'x'}S_2^x = \sin(\phi_2)\cos(\phi_2)\left(S_{21}^{yyS_2}S_{21}^{xxS_2} - S_{21}^{yyS_2}S_{21}^{xxS_2}\right) \tag{3.124}
\]

\[
S_{21\phi_2}^{x'y'}S_2^y = S_{21\phi_2}^{y'x'}S_2^x = \sin(\phi_2)\cos(\phi_2)\left(S_{21}^{yyMS_2S_2^{21}} - \frac{S_{21}^{xxMS_2S_2^{21}}}{S_{21}^{xxMTS_2^{21}}}\right) e^{-jK_0d} \tag{3.125}
\]

Following the same method for $S_{12}$

\[
S_{12\phi_2}^{x'y'}S_2^y = S_{12\phi_2}^{y'x'}S_2^x = \sin(\phi_2)\cos(\phi_2)\left(S_{12}^{yyS_2S_2^{12}} - S_{12}^{yyS_2S_2^{12}}\right) \tag{3.126}
\]

\[
S_{12\phi_2}^{x'y'}S_2^y = S_{12\phi_2}^{y'x'}S_2^x = \sin(\phi_2)\cos(\phi_2)\left(S_{12}^{yyMS_2S_2^{12}} - \frac{S_{12}^{xxMS_2S_2^{12}}}{S_{12}^{xxMTS_2^{12}}}\right) e^{-jK_0d} \tag{3.127}
\]

The biaxial sample is then rotated $\phi_2$ degrees and the following measurements are collected.

\[
S_{21}^{xyMS} = S_{21}^{yyA}S_{21\phi_2}^{x'y'}S_2^{xxB} \tag{3.128a}
\]

\[
S_{21}^{yxMS} = S_{21}^{xxA}S_{21\phi_2}^{y'x'}S_2^{yyB} \tag{3.128b}
\]

\[
S_{12}^{xyMS} = S_{12}^{yyB}S_{12\phi_2}^{x'y'}S_2^{xxA} \tag{3.128c}
\]

\[
S_{12}^{yxMS} = S_{12}^{xxB}S_{12\phi_2}^{y'x'}S_2^{yyA} \tag{3.128d}
\]

Solving for the A and B sections gives

\[
S_{21}^{yyA}S_{21}^{xxB} = \frac{S_{21}^{xyMS}}{S_{21\phi_2}^{x'y'}S_2^{xxB}} \tag{3.129a}
\]

\[
S_{21}^{xxA}S_{21}^{yyB} = \frac{S_{21}^{yxMS}}{S_{21\phi_2}^{y'x'}S_2^{yyB}} \tag{3.129b}
\]

\[
S_{12}^{yyB}S_{12}^{xxA} = \frac{S_{12}^{xyMS}}{S_{12\phi_2}^{x'y'}S_2^{xxA}} \tag{3.129c}
\]

\[
S_{12}^{xxB}S_{12}^{yyA} = \frac{S_{12}^{yxMS}}{S_{12\phi_2}^{y'x'}S_2^{yyA}} \tag{3.129d}
\]

where everything on the right side of the equations are now known. This can then be incorporated

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into Equation 3.118.

\[
S_{x'y'}S_{x'y'}S_{x'x'}S_{x'x'} = S_{xyMS}S_{xyMS}S_{x'y'S2}S_{x'x'S2}S_{21}S_{12}S_{21\phi2}S_{12\phi2}
\]

(3.130a)

\[
S_{y'x'}S_{y'x'}S_{y'x'}S_{y'x'} = S_{yxMS}S_{yxMS}S_{y'x'S2}S_{y'x'S2}S_{21}S_{12}S_{21\phi2}S_{12\phi2}
\]

(3.130b)

Isolating \(S_{21\phi}^{x'y'}\) and \(S_{21\phi}^{y'x'}\) gives the following:

\[
S_{x'y'}^{21\phi} = \sqrt{\frac{S_{xyMS}S_{xyMS}S_{x'y'S2}S_{x'y'S2}}{S_{xyMS}S_{xyMS}S_{21}S_{12}S_{21\phi2}S_{12\phi2}}} \exp \left\{ \frac{j}{2} \left( \frac{S_{xyMS}S_{xyMS}}{S_{xyMS}S_{21}} + \frac{S_{x'y'S2}S_{x'y'S2}}{S_{xyMS}S_{12}} \right) \right\}
\]

(3.131)

\[
S_{y'x'}^{21\phi} = \sqrt{\frac{S_{yxMS}S_{yxMS}S_{y'x'S2}S_{y'x'S2}}{S_{yxMS}S_{yxMS}S_{21}S_{12}S_{21\phi2}S_{12\phi2}}} \exp \left\{ \frac{j}{2} \left( \frac{S_{yxMS}S_{yxMS}}{S_{yxMS}S_{21}} + \frac{S_{y'x'S2}S_{y'x'S2}}{S_{yxMS}S_{12}} \right) \right\}
\]

(3.132)

Using Equation 3.131 and Equation 3.132, \(S_{21\phi}^{x'y'}\) and \(S_{21\phi}^{y'x'}\) can be isolated for the unknown sample with an unknown rotation without making any approximations.

### 3.3 Z Axis of Arbitrary Rotated Biaxial Sample

This section describes the process of calibrating the S parameters for a sample with unknown material parameter axial orientation around the Z axis and a known rotation around the Y axis (Figure 3.4).
3.3.1 X and Y axis

![Image of rotated axes](image)

Figure 3.4. Rotated Axis Around Z and Y Axis

Finding the \(\hat{z}\) parameters is very sensitive to error in the \(\hat{x}\) and \(\hat{y}\) parameters. Due to the Equivalency method in Section 3.2.3 possessing large fluctuations, the second sample method needs to be used as described in Section 3.2.4.

3.3.2 S Parameter Rotation

For a rotation around both the Z axis and the Y axis, a 3x3 matrix will need to be used to determine the new material parameters using the following equation

\[
\mathbf{S}_{\theta\phi} = \mathbf{T}_{zy} \mathbf{S} \mathbf{T}_{zy}^{-1}
\]

where \(\mathbf{T}_{zy}\) is a 3x3 matrix with the following relationship

\[
\mathbf{T}_{zy} = \begin{pmatrix}
\hat{x}' \cdot \hat{x} & \hat{x}' \cdot \hat{y} & \hat{x}' \cdot \hat{z} \\
\hat{y}' \cdot \hat{x} & \hat{y}' \cdot \hat{y} & \hat{y}' \cdot \hat{z} \\
\hat{z}' \cdot \hat{x} & \hat{z}' \cdot \hat{y} & \hat{z}' \cdot \hat{z}
\end{pmatrix} = \begin{pmatrix}
\cos(\phi)\cos(\theta) & \sin(\phi) & -\cos(\phi)\sin(\theta) \\
-\sin(\phi)\cos(\theta) & \cos(\phi) & \sin(\phi)\sin(\theta) \\
\sin(\theta) & 0 & \cos(\theta)
\end{pmatrix}
\]

(3.134)

the inverse equal to

\[
\mathbf{T}_{zy}^{-1} = \begin{pmatrix}
\hat{x}' \cdot \hat{x} & \hat{x}' \cdot \hat{y} & \hat{x}' \cdot \hat{z} \\
\hat{y}' \cdot \hat{x} & \hat{y}' \cdot \hat{y} & \hat{y}' \cdot \hat{z} \\
\hat{z}' \cdot \hat{x} & \hat{z}' \cdot \hat{y} & \hat{z}' \cdot \hat{z}
\end{pmatrix} = \begin{pmatrix}
\cos(\phi)\cos(\theta) & -\sin(\phi)\cos(\theta) & \sin(\theta) \\
\sin(\phi) & \cos(\phi) & 0 \\
-\cos(\phi)\sin(\theta) & \sin(\phi)\sin(\theta) & \cos(\theta)
\end{pmatrix}
\]

(3.135)

and the S parameters equal to
3.116 and 3.117 using the method described in Section 2.11.

The cross-polarization terms can be calibrated the same way as Equation 3.116 and 3.117 using the method described in Section 2.11.

\[
\begin{bmatrix}
S_{xx}^\prime \\
S_{yy}^\prime \\
S_{zz}^\prime
\end{bmatrix} = \begin{bmatrix}
S_{xx}^0 & 0 & 0 \\
0 & S_{yy}^0 & 0 \\
0 & 0 & S_{zz}^0
\end{bmatrix}
\]  

(3.136)

Substituting these into Equation 3.133 results in the following relationships:

\[
S_{b\phi}^{x',y'} = \cos^2(\theta) \cos^2(\phi) S_{xx}^\prime + \sin^2(\phi) S_{yy}^\prime + \sin^2(\theta) \cos^2(\phi) S_{zz}^\prime
\]  

(3.137a)

\[
S_{b\phi}^{y',x'} = -\cos^2(\theta) \sin(\phi) \cos(\phi) S_{xx}^\prime + \sin(\phi) \cos(\phi) S_{yy}^\prime - \cos(\phi) \sin(\phi) \sin^2(\theta) S_{zz}^\prime
\]  

(3.137b)

\[
S_{b\phi}^{x',x'} = \cos(\theta) \cos(\phi) \sin(\theta) S_{xx}^\prime - \cos(\phi) \sin(\theta) \cos(\theta) S_{zz}^\prime
\]  

(3.137c)

\[
S_{b\phi}^{y',y'} = -\cos^2(\theta) \cos(\phi) \sin(\phi) S_{xx}^\prime + \cos(\phi) \sin(\phi) S_{yy}^\prime - \sin^2(\theta) \cos(\phi) \sin(\phi) S_{zz}^\prime
\]  

(3.137d)

\[
S_{b\phi}^{y',x'} = \sin^2(\phi) \cos^2(\theta) S_{xx}^\prime + \cos^2(\phi) S_{yy}^\prime + \sin^2(\theta) \sin(\phi) S_{zz}^\prime
\]  

(3.137e)

\[
S_{b\phi}^{x',y'} = -\sin(\theta) \sin(\phi) \cos(\phi) S_{xx}^\prime + \sin(\phi) \cos(\phi) S_{zz}^\prime
\]  

(3.137f)

\[
S_{b\phi}^{x',x'} = \sin(\theta) \cos(\phi) \cos(\phi) S_{xx}^\prime - \sin(\phi) \cos(\phi) S_{zz}^\prime
\]  

(3.137g)

\[
S_{b\phi}^{y',y'} = -\cos(\theta) \sin(\phi) \sin(\theta) S_{xx}^\prime + \cos(\theta) \sin(\phi) \sin(\phi) S_{zz}^\prime
\]  

(3.137h)

\[
S_{b\phi}^{y',x'} = \sin^2(\theta) \cos^2(\phi) S_{xx}^\prime + \cos^2(\theta) S_{zz}^\prime
\]  

(3.137i)

### 3.3.3 Calibrate Z and Y Axis Rotated S Parameters

For \(S_{11b\phi}^{x'y'}\) and \(S_{11b\phi}^{y'z'}\), the path that the EM wave travels through the second sample is no longer the same as the distance the EM wave travels through the rotated sample. This means that the difference between the two needs to be taken into account when calibrating the S parameters as shown in Figure 3.5.
Figure 3.5. Second Sample Rotation S Parameter Representation

Where S represents the area of the unrotated biaxial sample and V represents the change in area when the unknown sample is rotated. This changes Equation 3.128 to

\[
\begin{align*}
S_{yy}^{MS2} &= S_{21}^{yy} S_{21}^{y^1 y^2} e^{-jK_0 V} S_{21}^{y^2 B} \\
S_{xx}^{MS2} &= S_{21}^{xx} S_{21}^{x^1 x^2} e^{-jK_0 V} S_{21}^{y^2 B}
\end{align*}
\] (3.140a)

(3.140b)

To find V, the difference between the sample diameter (d) and the new length caused by the rotation (l) needs to be found as shown in Figure 3.6.

\[
l = d \cos(\theta)
\] (3.141)

making V equal to the following.

\[
V = l - d = d(\sec(\theta) - 1)
\] (3.142)

Using Equation 3.142 and Equation 3.140, gives the following relationship

\[
\begin{align*}
S_{21}^{xxA} S_{21}^{yyB} &= S_{21}^{yy A} S_{21}^{y^1 y^2} e^{-jK_0 d(\sec(\theta) - 1)} \\
S_{21}^{yyA} S_{21}^{xxB} &= S_{21}^{xx A} S_{21}^{x^1 x^2} e^{-jK_0 d(\sec(\theta) - 1)}
\end{align*}
\] (3.143a)

(3.143b)

substituting Equation 3.143 into Equation 3.118

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\[ S'_{x'y'}S_{x'y'} = \frac{S_{xyMS}}{S_{x'y'S2}} \frac{S_{y'x'S2} S_{x'y'S2}}{S_{x'y'S2} S_{x'y'S2}} e^{-2jK_0 d(\sec \theta - 1)} \] (3.144a)

\[ S'_{y'x'}S_{y'x'} = \frac{S_{yxMS}}{S_{x'y'S2}} \frac{S_{y'x'S2} S_{y'x'S2}}{S_{x'y'S2} S_{x'y'S2}} e^{-2jK_0 d(\sec \theta - 1)} \] (3.144b)

and isolating \( S'_{21\phi} \) and \( S'_{21\phi} \) gives

\[
S'_{x'y'} = e^{-jK_0 d(\sec \theta - 1)} \sqrt{\frac{S_{xyMS}}{S_{x'y'S2}} \frac{S_{y'x'S2} S_{x'y'S2}}{S_{x'y'S2} S_{x'y'S2}}} \exp \left\{ \frac{j}{2} \left( \frac{S_{xyMS}}{S_{x'y'S2}} + \frac{S_{y'x'S2} S_{x'y'S2}}{S_{x'y'S2} S_{x'y'S2}} \right) \right\}
\] (3.145)

\[
S'_{y'x'} = e^{-jK_0 d(\sec \theta - 1)} \sqrt{\frac{S_{yxMS}}{S_{y'x'S2}} \frac{S_{y'x'S2} S_{y'x'S2}}{S_{y'x'S2} S_{y'x'S2}}} \exp \left\{ \frac{j}{2} \left( \frac{S_{yxMS}}{S_{y'x'S2}} + \frac{S_{y'x'S2} S_{y'x'S2}}{S_{y'x'S2} S_{y'x'S2}} \right) \right\}
\] (3.146)

### 3.3.4 Calculate Usable S Parameters

Since the method described in Section 3.1 uses the material parameters and S parameters that align with the X axis of the antenna, the values found from the Y and Z axis rotations need to be converted to match the same format. This can be done by first finding the equivalent S parameters that align with the antenna with the cross polarization removed using the following equations.

\[
S'_{x'x'} = S^{xx} \cos^2(\theta) + S^{yy} \sin^2(\theta)
\] (3.147a)

\[
S'_{y'y'} = S^{yy} \cos^2(\theta) + S^{xx} \sin^2(\theta)
\] (3.147b)

where \( S'_{x'x'} \) represents the S parameter for the rotated X axis without any cross polarization and \( S'_{y'y'} \) represents the S parameter for the rotated Y axis without any cross polarization.

Next, the perceived material parameters that line up with the antenna orientation are calculated.
\[ \epsilon_{x',x'}^\phi = \epsilon_{xx} \cos^2(\theta) + \epsilon_{yy} \sin^2(\theta) \quad (3.148a) \]
\[ \epsilon_{y',y'}^\phi = \epsilon_{yy} \cos^2(\theta) + \epsilon_{xx} \sin^2(\theta) \quad (3.148b) \]
\[ \mu_{x',x'}^\phi = \mu_{xx} \cos^2(\theta) + \mu_{yy} \sin^2(\theta) \quad (3.148c) \]
\[ \mu_{y',y'}^\phi = \mu_{yy} \cos^2(\theta) + \mu_{xx} \sin^2(\theta) \quad (3.148d) \]

Equation 3.147 and Equation 3.148 can then be used for the method outlined in Section 3.1 to obtain the zz material parameters.
4. Methodology

Using the derivations from Section 3 along with the background provided in Section 2, the method for collecting and processing the necessary data to extract the material parameters will be discussed in this section.

4.1 Experimental Setup

In this research project a Georgia Tech Research Institute FBS, depicted in Figure 4.1, was utilized to collect S parameters. It is composed of an open boundary quad ridge horn antenna (Figure 4.2), a FBS coaxial switch (Figure 4.3), a collimating lens (Figure 4.4a), and a sample holder (Figure 4.4b). The antenna possesses the ability to have either horizontal or vertical electric fields emitted and is controlled using the coaxial switches. The left switch controls antenna 1 (left side of Figure 4.1) and the right switch controls antenna 2 (right side of Figure 4.1). By having the switch in the up position towards the H, that antenna is set to emit and receive electric fields in the horizontal/\(\hat{y}\) direction. The down position, towards the V, similarly sets that antenna to emit and receive electric fields in the vertical/\(\hat{x}\) direction. Changing the settings of the coaxial switches, the type of S parameters collected and emitted can be controlled [12].

The sample holder possesses two degrees of rotation, one around the X axis, and the other around the Y axis. This allows the sample to be rotated during measurements.

The antennas are connected to a PNA depicted in Figure 4.5a. The PNA controls which frequencies are emitted by the antennas and collects the resulting S parameters. The \(S_{21}, S_{12}, S_{11}, S_{22}\) parameters are all collected as depicted in Figure 4.5b, showing the measured S.
parameters displayed on the PNA.

4.1.1 Testing Sample

For this research project, biaxial, non-magnetic samples were provided by Dr. Collins (Figure 4.6). This allows the relative permeability of the sample to be set to 1 requiring the need of only $S_{21}$ and $S_{12}$ measurements shown in Figure 4.6. This sample possesses a uniform honeycomb design giving it biaxial properties that are uniform for each axis across the entire sample. Unfortunately, the samples available were too small to fit in the wave-guide holder without the frame covering a large portion of the available sample (1/3 of the total sample). As a result, a larger isotropic sample was required to be used to hold the biaxial sample in place during testing (figure 4.7) and placed in the FBS (Figure 4.8).

Due to the small sample, the focus beam spot size exceeds the area of the sample for frequencies below 15 GHz causing fringing effects, this restricts the usable frequencies for testing to 15-18 GHz.
4.2 Biaxial Sample with Oblique Incidence Z Axis

Here the process of finding the material parameter for all three axes of a biaxial, non-magnetic sample aligned with the antennas is discussed.

4.2.1 Data Collection

All the S parameters collected are first gated using the method described in Section 2.9. The first set of data that needs to be collected are the through S parameters without a sample in the FBS with \( d \) equal to the sample/s total thickness.

\[
S_{21}^{xx MT} = S_{21}^{xx A} e^{-jK_0 d} S_{21}^{xx B} \\
S_{21}^{yy MT} = S_{21}^{yy A} e^{-jK_0 d} S_{21}^{yy B} \\
S_{12}^{xx MT} = S_{12}^{xx B} e^{-jK_0 d} S_{12}^{xx A} \\
S_{12}^{yy MT} = S_{12}^{yy B} e^{-jK_0 d} S_{12}^{yy A}
\]

Next, the isotropic sample (subscript \( i \)) will need to be measured. Since it is isotropic (Verified in
The through and isotropic measurements will be needed for all of the methods described in this thesis.

The biaxial sample will also need to be measured in conjunction with the isotropic sample.
(a) Sample 1

(b) Sample 2

Figure 4.6. Biaxial Samples

Figure 4.7. Biaxial Sample Connected With Isotropic Sample

(subscript $i u$) as shown in Figure 4.8

\[
S_{21i u}^{xxMS} = S_{21}^{xxA} S_{21i u}^{xx} S_{21}^{xxB}
\]

\[
S_{21i u}^{yyMS} = S_{21}^{yyA} S_{21i u}^{yy} S_{21}^{yyB}
\]  \hspace{1cm} (4.3a)

\[
S_{12i u}^{xxMS} = S_{12}^{xxA} S_{12i u}^{xx} S_{12}^{xxA}
\]

\[
S_{12i u}^{yyMS} = S_{12}^{yyA} S_{12i u}^{yy} S_{12}^{yyA}
\]  \hspace{1cm} (4.3b)

Lastly, the sample needs to be rotated $\theta$ degrees around the Y axis as shown in Figure 4.9, and the
4.2.2 Experimental Data

To calibrate the data, first the through data is used to removed the $S^A$ and $S^B$ parameters. Next, the $S_{21}$ and $S_{12}$ parameters are multiplied together to try and average out any errors associated with one antenna over the other. Lastly, the $S_{21}$ value is isolated. This results in the following equation for both the X orientated $S$ parameters and the Y orientated $S$ parameters.

\[
\begin{align*}
S_{x MS}^{21, iu} &= S_{21}^A S_{21}^S S_{21}^B \\
S_{x MS}^{12, iu} &= S_{12}^{S} S_{12}^A S_{12}^S
\end{align*}
\] (4.4a)

\[
\begin{align*}
S_{x MS}^{21, iu} &= S_{12}^{M T} S_{12}^A S_{12}^S \\
S_{x MS}^{12, iu} &= S_{21}^{M T} S_{21}^A S_{21}^S
\end{align*}
\] (4.4b)

\[
\begin{align*}
S_{x x S}^{21 iu} &= S_{12 iu}^S = \sqrt{S_{x x S}^{21 iu} S_{x x S}^{12 iu}} = e^{-jK_0(d_i + d_u)} \sqrt{\frac{S_{x x MS}^{21 iu}}{S_{21}^{M T}}} \exp \left\{ \frac{j}{2} \left( \frac{S_{x x MS}^{21 iu}}{S_{21}^{M T}} + \frac{S_{x x MS}^{12 iu}}{S_{12}^{M T}} \right) \right\}
\end{align*}
\] (4.5)
Figure 4.9. Biaxial Sample on Isotropic Sample Rotated Around Y Axis

\[ S_{y1y}^{y} = S_{12}^{y} = \sqrt{S_{y1y}^{y} S_{y1y}^{y}} = e^{-jK_0(d_i + d_u)} \left[ \frac{S_{y1y}^{y} S_{12}^{y}}{S_{y1y}^{y}} \right] \exp \left\{ \frac{j}{2} \left( \frac{\varphi_{y1y}^{y} S_{12}^{y}}{S_{12}^{y}} + \frac{\varphi_{12}^{y} S_{y1y}^{y}}{S_{y1y}^{y}} \right) \right\} \]

(4.6)

The Isotropic sample follows the same process but only needs one axis.

\[ S_{x1x}^{x} = S_{12}^{x} = \sqrt{S_{x1x}^{x} S_{12}^{x}} = e^{-jK_0(d_i)} \left[ \frac{S_{x1x}^{x} S_{12}^{x}}{S_{x1x}^{x}} \right] \exp \left\{ \frac{j}{2} \left( \frac{\varphi_{x1x}^{x} S_{12}^{x}}{S_{12}^{x}} + \frac{\varphi_{12}^{x} S_{x1x}^{x}}{S_{x1x}^{x}} \right) \right\} \]

(4.7)

For the rotated Sample, the internal angle first needs to be calculated using Snell’s law described in Section 2.12 for both the isotropic sample

\[ \theta_i = \arcsin \left( \frac{\sin(\theta)}{\sqrt{\mu_{ir} \varepsilon_{ir}}} \right) \]

(4.8)

and the biaxial sample.

\[ \theta_u = \arcsin \left( \frac{\sin(\theta)}{\sqrt{\mu_{ur} \varepsilon_{ur}}} \right) \]

(4.9)

Since the isotropic sample has the same material parameters for all axes, the permittivity can be calculated first without using non-oblique measurements and inserted into this equation. However, for the biaxial sample, the perceived permittivity of the sample needs to be calculated as follows.

4-7
\[ \epsilon_{ur} = \frac{\epsilon_{xx} \epsilon_{zz}}{\cos^2(\theta_u) \epsilon_{zz} + \sin^2(\theta_u) \epsilon_{xx}} \] (4.10)

For \( \epsilon_{yy} \) and \( \epsilon_{xx} \), \( \theta \) can be set to zero and calculated before the oblique calculations. That leaves the only unknown as \( \epsilon_{zz} \). To solve for \( \epsilon_{zz} \), an initial guess will need to be made first allowing the following calibration to be solved.

\[ S_{21\theta,iu}^x = e^{-jK_0(d_i \cos(\theta_i) + d_u \cos(\theta_u))} \left[ \frac{S_{xxMS}^x}{S_{21}^x} \right] \frac{\lambda}{\lambda_{12}^x} \exp \left\{ \frac{j}{2} \left( \frac{S_{xxMS}^x}{S_{21}^x} + \frac{S_{xxMS}^y}{S_{12}^y} \right) \right\} \] (4.11)

After the solution for \( \epsilon_{zz} \) is obtained, it will need to be inserted back into these equations and solved again in a loop. This will be done until the difference between the old and new value is less than a predetermined error value.

### 4.2.3 Theoretical Solution

The next step is to find the theoretical solution to compare the experimental against for calculating permittivity. First the propagation constant is found.

\[ K^\parallel = \pm \sqrt{\frac{\omega^2 \epsilon_{xx} \epsilon_{zz} \mu_{yy}}{\cos^2(\theta) \epsilon_{zz} + \sin^2(\theta) \epsilon_{xx}}} \] (4.12)

Next, the the impedance of the sample is calculated

\[ \eta^\parallel = \frac{K^\parallel \cos(\theta)}{\omega \epsilon_{yy}} \] (4.13)

From there, solve for \( P \) and \( R \)

\[ P^\parallel = e^{-jK^\parallel \cos(\theta)} \quad R^\parallel = \frac{\eta^\parallel - 1}{\eta^\parallel + 1} \] (4.14)

\( P \) and \( R \) can then be used to solve for \( S_{21} \) theoretical.

\[ S_{21}^{theoretical} = \frac{P^\parallel (1 - R^2 \parallel)}{1 - P^2 \parallel R^2 \parallel} \] (4.15)

This will first need to be done using non-oblique incidence, to do this, set \( \theta \) equal to 0. for \( \epsilon_{yy} \) change \( \epsilon_{xx} \) and \( \epsilon_{yy} \) with each other in the equations. For Isotropic, Use \( \theta_i \) for \( \theta \), and for biaxial use
\( \theta_u \) for \( \theta \).

Once this has been calculated, the \( S_{21}^{thy} \) for the biaxial and isotropic samples combined need to be found. This can be done using A matrix transformations described in Section 2.6.

\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix} = \frac{1}{S_{21}} \begin{pmatrix}
1 & -S_{22} \\
S_{11} & S_{12}S_{21} - S_{11}S_{22}
\end{pmatrix}
\] (4.16)

Using the above equation to transform \( S_{21}^{thy} \) for both the isotropic and biaxial sample the total A matrix can be found.

\[ A^\text{total} = A^{iso} A^{biax} \] (4.17)

Next, the total A matrix is transformed back into S parameters

\[
\begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix} = \frac{1}{A_{11}} \begin{pmatrix}
A_{21} & A_{11}A_{22} - A_{21}A_{12} \\
1 & -A_{12}
\end{pmatrix}
\] (4.18)

When calculating \( S_{21}^\text{total} \), these equations simplify to the following.

\[ A^{total}_{11} = A^{iso}_{11} A^{biax}_{11} + A^{iso}_{12} A^{biax}_{21} \] (4.19)

\[ S_{21}^\text{total} = \frac{1}{A^{total}_{11}} = S_{21}^{thy} \] (4.20)

Lastly, the difference between \( S_{21}^{thy} \) and \( S_{21}^{exp} \) can be minimized using an iterative method with the unknown being \( \epsilon_{xx} \) or \( \epsilon_{yy} \) for the non-oblique incidence and \( \epsilon_{zz} \) for oblique incidence in the \( K \) calculation.

\[ \left\| S_{21}^{thy} - S_{21}^{exp} \right\|_2 < \text{Error} \approx 0 \] (4.21)

### 4.3 Rotated Biaxial Sample Using Equivalency Method

The process used to extract the material parameters of an arbitrarily rotated biaxial sample using the method in Section 3.2.3 will be discussed in this section.
4.3.1 Data Collection

For this method, the $S_{21iu}^{xxMS}$, $S_{21iu}^{yyMS}$, $S_{21iu}^{yxMS}$ and $S_{21iu}^{xyMS}$ parameters for the biaxial and isotropic samples combined will need to be collected.

\begin{align}
S_{21iu}^{xxMS} &= S_{21}^{xxA} S_{21}^{x'x'} S_{21}^{xxB} \\
S_{21iu}^{yyMS} &= S_{21}^{yyA} S_{21}^{y'y'} S_{21}^{yyB} \\
S_{12iu}^{xxMS} &= S_{12}^{xxB} S_{12}^{x'x'} S_{12}^{xxA} \\
S_{12iu}^{yyMS} &= S_{12}^{yyB} S_{12}^{y'y'} S_{12}^{yyA} \\
S_{21iu}^{xyMS} &= S_{21}^{xyA} S_{21}^{x'y'} S_{21}^{xyB} \\
S_{21iu}^{yxMS} &= S_{21}^{yxA} S_{21}^{y'x'} S_{21}^{yxB}
\end{align}

(4.22a) \hspace{1cm} (4.22b) \hspace{1cm} (4.22c) \hspace{1cm} (4.22d)

In addition to these, the cross polarization $S$ parameters will also need to be collected.

\begin{align}
S_{21iu}^{xyMS} &= S_{21}^{xyA} S_{21}^{x'y'} S_{21}^{xyB} \\
S_{21iu}^{yxMS} &= S_{21}^{yxA} S_{21}^{y'x'} S_{21}^{yxB} \\
S_{12iu}^{xyMS} &= S_{12}^{xyB} S_{12}^{y'x'} S_{12}^{xyA} \\
S_{12iu}^{yxMS} &= S_{12}^{yxB} S_{12}^{x'y'} S_{12}^{yxA}
\end{align}

(4.24)

These measurements were collected with the sample rotated around the Z axis as shown in Figure 4.10.
4.3.2 Calibration

The same method described previously can be used to calibrate the $S^{xx}$ and $S^{yy}$ parameters. To calibrate the cross-polarization terms, $S_{21}$ and $S_{12}$ need to be combined.

\[
\begin{align*}
S_{21}^{x'y'}S_{12}^{x'y'} &= \frac{S_{xy}^MS_{xy}^MM}{S_{21}^{21u}S_{12}^{12u}} \\
S_{21}^{y'z'}S_{12}^{y'z'} &= \frac{S_{yx}^MS_{yx}^MM}{S_{21}^{21u}S_{12}^{12u}}
\end{align*}
\]  

(4.25)

Next, the following approximation is made for free space.

\[
S^{xx} = S^{yy}
\]  

(4.27)

Since Section A and Section B are both free space, this equivalency can be used to generate the following.

\[
\begin{align*}
S_{21}^{x'y'}S_{12}^{x'y'} &= \sqrt{S_{xy}^MS_{xy}^MM} = \frac{S_{xy}^MS_{xy}^MM}{S_{21}^{21u}S_{12}^{12u}} \exp \left\{ \frac{j}{2} \left( \frac{S_{xy}^MS_{xy}^MM}{S_{21}^{21u}S_{12}^{12u}} + \frac{S_{yx}^MS_{yx}^MM}{S_{21}^{21u}S_{12}^{12u}} \right) \right\} \\
S_{21}^{y'z'}S_{12}^{y'z'} &= \sqrt{S_{yx}^MS_{yx}^MM} = \frac{S_{yx}^MS_{yx}^MM}{S_{21}^{21u}S_{12}^{12u}} \exp \left\{ \frac{j}{2} \left( \frac{S_{yx}^MS_{yx}^MM}{S_{21}^{21u}S_{12}^{12u}} + \frac{S_{xy}^MS_{xy}^MM}{S_{21}^{21u}S_{12}^{12u}} \right) \right\}
\end{align*}
\]  

(4.28)

Solving for $S_{21}$ gives the calibration equations.

\[
\begin{align*}
S_{21}^{x'y'} &= \sqrt{S_{21}^{x'y'}S_{12}^{x'y'}} = e^{-jK_0(di+du)} \sqrt{S_{xy}^MS_{xy}^MM} \left[ \frac{S_{xy}^MS_{xy}^MM}{S_{21}^{21u}S_{12}^{12u}} \right] \exp \left\{ \frac{j}{2} \left( \frac{S_{xy}^MS_{xy}^MM}{S_{21}^{21u}S_{12}^{12u}} + \frac{S_{yx}^MS_{yx}^MM}{S_{21}^{21u}S_{12}^{12u}} \right) \right\} \\
S_{21}^{y'z'} &= \sqrt{S_{21}^{y'z'}S_{12}^{y'z'}} = e^{-jK_0(di+du)} \sqrt{S_{yx}^MS_{yx}^MM} \left[ \frac{S_{yx}^MS_{yx}^MM}{S_{21}^{21u}S_{12}^{12u}} \right] \exp \left\{ \frac{j}{2} \left( \frac{S_{yx}^MS_{yx}^MM}{S_{21}^{21u}S_{12}^{12u}} + \frac{S_{xy}^MS_{xy}^MM}{S_{21}^{21u}S_{12}^{12u}} \right) \right\}
\end{align*}
\]  

(4.30)

4.3.3 Align S Parameters

Using Equation 3.100, the S parameters correlating to the Axes of the biaxial sample can be found. Since biaxial samples do not posses any cross-polarization when aligned with the antenna, the
Aligned cross polarization terms can be set to zero to solve for the angle of rotation.

\[
S_{21}^{\text{xy}} = \cos(\phi)\sin(\phi)(S_{21\phi}^{x'x'S} - S_{21\phi}^{x'x'S} + \cos^2(\phi)S_{21\phi}^{y'y'S} - S^2(\phi)S_{21\phi}^{y'y'}) = 0 \quad (4.32)
\]

\[
S_{21}^{\text{yx}} = \cos(\phi)\sin(\phi)(S_{21\phi}^{x'x'S} - S_{21\phi}^{x'x'S} + \cos^2(\phi)S_{21\phi}^{y'y'S} - S^2(\phi)S_{21\phi}^{y'y'}) = 0 \quad (4.33)
\]

After the angle has been calculated, the non-rotated S parameters correlating to the X and Y axes of the sample can be calculated using the following.

\[
S_{21}^{\text{xxS}} = \cos^2(\phi)S_{21\phi}^{x'x'S} - \sin(\phi)\cos(\phi)(S_{21\phi}^{y'y'S} + S_{21\phi}^{y'y'S}) + \sin^2(\phi)S_{21\phi}^{y'y'S} \quad (4.34)
\]

\[
S_{21}^{\text{yyS}} = \sin^2(\phi)S_{21\phi}^{x'x'S} - \sin(\phi)\cos(\phi)(S_{21\phi}^{y'y'S} + S_{21\phi}^{y'y'S}) + \cos^2(\phi)S_{21\phi}^{y'y'S} \quad (4.35)
\]

With the aligned S parameters, the solution can be obtained the same way as described previously in Section 4.2.

### 4.4 Rotated Biaxial Sample Using the 2nd Sample Method

The process used to extract the material parameters of an arbitrarily rotated biaxial sample using the method in Section 3.2.4 is described in this section.

#### 4.4.1 Data Collection

The same data collected in Section 4.3.1 is needed, as well as the following for a second sample with known material parameter axial orientation and be aligned with the antenna.

\[
S_{21\phi}^{\text{xxA}}S_{21\phi}^{\text{xxS}}S_{21\phi}^{\text{xxB}} = S_{21\phi}^{\text{xxA}}S_{21\phi}^{\text{xyA}}S_{21\phi}^{\text{xyB}} \quad (4.36a)
\]

\[
S_{21\phi}^{\text{yyA}}S_{21\phi}^{\text{yyS}}S_{21\phi}^{\text{yyB}} = S_{21\phi}^{\text{yyA}}S_{21\phi}^{\text{yyS}}S_{21\phi}^{\text{yyB}} \quad (4.36b)
\]

Then the second sample is rotated $\phi_2$ degrees and the following S parameters collected:
The unrotated S parameters first have the A and B portions removed as before.

\[
S_{xx}^{21} = \frac{S_{21}^{xxMS} - jK_0(d_a + d_i)}{S_{21}^{xxMT} e^{jK_0(d_a + d_i)}}
\]

(4.38)

\[
S_{yy}^{21} = \frac{S_{21}^{yyMS} - jK_0(d_a + d_i)}{S_{21}^{yyMT} e^{jK_0(d_a + d_i)}}
\]

(4.39)

Taking Equation 3.97, the rotated S parameter can be calculated.

\[
S_{21}^{xyMS} = \frac{S_{21}^{xxMS}}{S_{21}^{xxMS}} = \frac{S_{21}^{yyMS}}{S_{21}^{yyMS}}
\]

(4.40)

Combining Equation 4.37 and 4.40 results in the relationship between xx and yy S parameters for A and B.

\[
S_{21}^{yyA} S_{21}^{xxB} = S_{12}^{xyMS} S_{12}^{xxB} = S_{12}^{yyA} S_{12}^{xxB} = S_{12}^{yyA} S_{12}^{xxB}
\]

(4.41a)

(4.41b)

(4.41c)

(4.41d)

Using these relationships, Equation 4.25 and Equation 4.26 can be solved directly resulting in the following:
Lastly, isolating $S_{21}$ gives:

$$S_{21} = \sqrt{\frac{S_{xyMS} S_{yxMS}}{S_{21iu} S_{12iu}}} \exp \left\{ \frac{j}{2} \left( \frac{S_{xyMS} S_{xyMS}}{S_{21iu} S_{12iu}} + \angle \frac{S_{xyMS} S_{xyMS}}{S_{21iu} S_{12iu}} \right) \right\}$$

Next, the S parameters aligned with the antenna can be obtained the same way as in Section 4.3.3.

### 4.5 Biaxial Sample with Oblique Incidence and Unknown Z Axis Rotation

This section describes the process used to extract the Z axis material parameters for a non-magnetic biaxial sample with unknown material parameter axial orientation around the Z axis and a known rotation around the Y axis using the method described in Section 3.3. This section will assume that the process detailed in Section 4.4 has already taken place.
4.5.1 Data Collection

positioning the unknown sample as shown in Figure 4.11, the following measurements need to be collected.

\[
S_{zz MS}^{21u} = S_{21}^{zz} A_{zz}^{x'} z' S_{zz} B_{zz}^{21} \tag{4.46a}
\]
\[
S_{yy MS}^{21u} = S_{21}^{yy} A_{yy}^{y'} y' S_{yy} B_{yy}^{21} \tag{4.46b}
\]
\[
S_{zz MS}^{12u} = S_{12}^{zz} A_{zz}^{x'} z' S_{zz} A_{zz}^{12} \tag{4.46c}
\]
\[
S_{yy MS}^{21u} = S_{21}^{yy} A_{yy}^{y'} y' S_{yy} A_{yy}^{21} \tag{4.46d}
\]
\[
S_{xx MS}^{21u} = S_{21}^{xx} A_{xx}^{x'} x' S_{xx} B_{xx}^{21} \tag{4.46e}
\]
\[
S_{yx MS}^{21u} = S_{21}^{yx} A_{yx}^{x'} y' S_{yx} A_{yx}^{21} \tag{4.46f}
\]
\[
S_{yx MS}^{12u} = S_{12}^{yx} A_{yx}^{x'} y' S_{yx} A_{yx}^{12} \tag{4.46g}
\]
\[
S_{xx MS}^{12u} = S_{12}^{xx} A_{xx}^{x'} x' S_{xx} A_{xx}^{12} \tag{4.46h}
\]

4.5.2 Calibration

to calibrate the rotated sample using the second sample method, the phase shift for the second sample needs to be modified so that it matches the new increase in travel distance within the sample caused by the Y axis rotation. This differences is equal to V:
\[ V = l - d = d(\sec(\theta) - 1) \quad (4.47) \]

Applying this to Equation 4.41 gives:

\[
S_{21}^{yyA} S_{21}^{xxB} = \frac{S_{xyMS}^{21iu}}{S_{21}^{21iu}} e^{-jK_0(\sec(\theta_0) - 1) + du(\sec(\theta_a) - 1)} \quad (4.48a)
\]

\[
S_{21}^{xxA} S_{21}^{yyB} = \frac{S_{zyMS}^{21iu}}{S_{21}^{21iu}} e^{-jK_0(\sec(\theta_0) - 1) + du(\sec(\theta_a) - 1)} \quad (4.48b)
\]

\[
S_{12}^{yyA} S_{12}^{xxB} = \frac{S_{zyMS}^{12iu}}{S_{12}^{12iu}} e^{-jK_0(\sec(\theta_0) - 1) + du(\sec(\theta_a) - 1)} \quad (4.48c)
\]

\[
S_{12}^{xxA} S_{12}^{yyB} = \frac{S_{xyMS}^{12iu}}{S_{12}^{12iu}} e^{-jK_0(\sec(\theta_0) - 1) + du(\sec(\theta_a) - 1)} \quad (4.48d)
\]

Combining the \( S_{21} \) and \( S_{12} \) cross-polarization terms with the above equations leads to:

\[
S_{21\phi}^{x'y'S} S_{12\phi}^{x'y'S} = \frac{S_{xyMS}^{21iu}}{S_{21}^{21iu}} \frac{S_{xyMS}^{12iu}}{S_{12}^{12iu}} \frac{S_{zyMS}^{21iu}}{S_{21}^{21iu}} \frac{S_{zyMS}^{12iu}}{S_{12}^{12iu}} e^{-2jK_0(\sec(\theta_0) - 1) + du(\sec(\theta_a) - 1)} \quad (4.49)
\]

\[
S_{21\phi}^{y'x'S} S_{12\phi}^{y'x'S} = \frac{S_{zyMS}^{21iu}}{S_{21}^{21iu}} \frac{S_{zyMS}^{12iu}}{S_{12}^{12iu}} \frac{S_{xyMS}^{21iu}}{S_{21}^{21iu}} \frac{S_{xyMS}^{12iu}}{S_{12}^{12iu}} e^{-2jK_0(\sec(\theta_0) - 1) + du(\sec(\theta_a) - 1)} \quad (4.50)
\]

Isolating \( S_{21} \) gives:

\[
S_{21\phi}^{x'y'S} = e^{-jK_0(\sec(\theta_0) - 1) + du(\sec(\theta_a) - 1)} \begin{bmatrix} S_{xyMS}^{21iu} & S_{xyMS}^{12iu} \end{bmatrix} \begin{bmatrix} S_{zyMS}^{21iu} & S_{zyMS}^{12iu} \end{bmatrix}^{-1} \begin{bmatrix} S_{xyMS}^{21iu} & S_{xyMS}^{12iu} \end{bmatrix} \begin{bmatrix} S_{zyMS}^{21iu} & S_{zyMS}^{12iu} \end{bmatrix}^{-1} \]

\[
\exp \left( \frac{j}{2} \left( \frac{S_{xyMS}^{21iu}}{S_{xyMS}^{21iu}} + \frac{S_{xyMS}^{12iu}}{S_{xyMS}^{12iu}} \right) \right) \quad (4.51)
\]
\[ S_{21\theta\phi}^{y'x'S} = e^{-jK_0(di(\sec(\theta)) - 1) + du(\sec(\theta_u) - 1))} \sqrt{\frac{S_{yzMS}S_{yzMS}}{S_{21u}^{yzMS2}} - \frac{S_{12u}^{yzMS2}}{S_{21u}^{yzMS2}}} \]

\[ \exp\left\{ \frac{j}{2} \left( \frac{S_{yzMS}S_{yzMS}}{S_{21u}^{yzMS2}} - \frac{S_{12u}^{yzMS2}}{S_{21u}^{yzMS2}} \right) \right\} \] (4.52)

Next, the non-rotated S parameters are calculated using the method from Section 4.3.3 but using the same \( \phi \) that was calculated previously.

Next, the rotated S parameters need to be calculated with the cross polarization terms removed.

\[ S_{21\theta\phi}^{y'x'S} = S_{21\theta\phi}^{xx} \cos^2 \theta + S_{21\theta\phi}^{yy} \sin^2 \theta \] (4.53)

The rotated Permittivity also needs to be calculated from the permittivity found from the non-oblique incidence calculations.

\[ \epsilon_{x'x'} = \epsilon_{xx} \cos^2 \theta + \epsilon_{yy} \sin^2 \theta \] (4.54)

These two terms can then be inserted into the equations from Section 4.2 and solved the same way.
5. Error Analysis

This section goes into detail over the errors involved with the methods described previously and with the experimental setup.

5.1 Setup Error

One of the major sources of error for this experiment was the gaussian beam depth of the focus region where the field can be considered planar. Since all of the calculations assume that the EM field passing through the sample is a plane wave, any deviation induces error. When oblique incidence is introduced, this problem is further increased due to the wave hitting the boundary layer of the sample causes refraction. Another source of error in the plane wave assumption is operating outside of the 11 GHz that the collimating lens was designed for. Though this error is less severe it still causes increased fluctuations in the data as the frequency gets higher and higher.

Another source of error is due to background radiation. As the FBS is an open system, the EM radiation that exists within the room also interacts the the setup and are received by the antennas. This background radiation is mostly removed when the data is gated.

5.2 Analytical Error Analysis

The following calculations show the mathematical relationship between measurement uncertainty and the resulting solution.

5.2.1 Oblique Incidence Error

The measurement errors associated with the method of extracting the Z material parameters are the thickness of the biaxial sample, thickness of the isotropic sample, angle that the sample is rotated about the Y axis and the assumption that the antenna and sample axis align. These errors
leads to the following terms:

\[ d_u^e = d_u + \delta_d \]  
\[ d_i^e = d_i + \delta_d \]  
\[ \theta^e = \theta + \delta_\theta \]  
\[ \phi^e = \delta_\phi \]  

where \( \delta_d \) is the error in the biaxial sample width measurement, \( \delta_d \) is the error in the isotropic sample width measurement, \( \delta_\theta \) is the error in the rotation around the Y axis measurement, and \( \delta_\phi \) is the error in the assumption that the sample is aligned with the antenna.

### 5.2.1.1 \( \hat{x} \) and \( \hat{y} \) Error

First, the \( S_{21} \) measurement Equation 2.45 changes in the following way

\[ S_{21}^{MSe} = \frac{S_{21}^{MS}}{S_{21}^{MT}} e^{-jK_0(d_u + \delta_d + d_i + \delta_d)} \]  

(5.2)

Comparing the error term with no error gives

\[ S_{21}^{MSe} = S_{21}^{MS} e^{-jK_0(\delta_d + \delta_d)} \]  

(5.3)

Next, the error associated with not having the antenna perfectly aligned with the sample is included using Equation 3.97, changing Equation 5.3 to the following.

\[ S_{21}^{MSe} = (S_{21}^{xMS} \cos^2(\delta_\theta) + S_{21}^{yMS} \sin^2(\delta_\theta)) e^{-jK_0(\delta_d + \delta_d)} \]  

(5.4)

This gives the error associated with the measurement/experimental term.

To find the theoretical error, Equation 3.60 is modified to include measurement error. \( P^e \) can then be separated into the true P value multiplied by an error term

\[ P^e = e^{-jK_0(d_u + \delta_d)} = P e^{-jK_0 \delta_d} \]  

(5.5)

substituting \( P^e \) into Equation 3.73 gives the following:

5-2
\[ S_{21}^e = \frac{P e^{-jK_0 \delta_d} (1 - R^2)}{1 - P^2 R^2 e^{-2jK_0 \delta_d}} \]  \hspace{1cm} (5.6)

Setting

\[ \delta_{S_{21}} = \frac{(1 - R^2 P^2) e^{-jK_0 \delta_d}}{1 - P^2 R^2 e^{-2jK_0 \delta_d}} \]  \hspace{1cm} (5.7)

Equation 5.6 can be simplified to

\[ S_{21}^e = S_{21} \delta_{S_{21}} \]  \hspace{1cm} (5.8)

where \( \delta_{S_{21}} \) is the multiplicative error associated with \( S_{21} \) theoretical.

Equation 5.9 is found by taking error into account while determining \( S_{11} \).

\[ S_{11}^e = \frac{R(1 - P^2 e^{-2jK_0 \delta_d})}{1 - P^2 R^2 e^{-jK_0 \delta_d}} \]  \hspace{1cm} (5.9)

Setting

\[ \delta_{S_{11}} = \frac{(1 - P^2 e^{-2jK_0 \delta_d})(1 - P^2 R^2)}{(1 - P^2 R^2 e^{-2jK_0 \delta_d})(1 - P^2)} \]  \hspace{1cm} (5.10)

gives the multiplicative error associated with \( S_{11} \).

\[ S_{11}^e = S_{11} \delta_{S_{11}} \]  \hspace{1cm} (5.11)

The A matrix errors can then be found from Equation 2.31

\[ A_{11}^e = \frac{1}{S_{21}^e} = \frac{1}{S_{21} \delta_{S_{21}}} \]  \hspace{1cm} (5.12a)

\[ A_{22}^e = \frac{s_{21}^e s_{12}^e - s_{11}^e s_{22}^e}{S_{21}^e} = \frac{S_{21} S_{12} \delta_{S_{21}}^2 - S_{11} S_{22} \delta_{S_{11}}^2}{S_{21} \delta_{S_{21}}} \]  \hspace{1cm} (5.12b)

\[ A_{12}^e = -\frac{s_{22}^e}{S_{21}^e} = -\frac{S_{22} \delta_{S_{11}}}{S_{21} \delta_{S_{21}}} \]  \hspace{1cm} (5.12c)

\[ A_{21}^e = \frac{s_{11}^e}{S_{21}^e} = \frac{-S_{22} \delta_{S_{11}}}{S_{21} \delta_{S_{21}}} \]  \hspace{1cm} (5.12d)

Using the A matrix errors for both the isotropic sample and biaxial sample, the total error in the theoretical \( S_{21} \) parameters can be found using Equation 2.36 with subscript \( b \) representing biaxial,
subscript i representing isotropic and subscript t representing the two combined.

\[ A_{11t}^e = A_{11i}^e A_{11b}^e + A_{12i}^e A_{21b}^e = \frac{A_{11i} A_{11b}}{\delta S_{21i} \delta S_{21b}} + \frac{A_{12i} A_{21b} \delta S_{11} \delta S_{11b}}{\delta S_{21i} \delta S_{21b}} \]  \hspace{1cm} (5.13)

Setting

\[ \delta_{At} = \frac{A_{11i} A_{11b} + A_{12i} A_{21b} \delta S_{11} \delta S_{11b}}{A_{11i} \delta S_{21i} \delta S_{21b}} \]  \hspace{1cm} (5.14)

gives the following:

\[ A_{11t}^e = A_{11i} \delta_{At} \]  \hspace{1cm} (5.15)

Equation 2.36 combined with Equation 5.15 then gives the theoretical \( S_{21} \) value equal to

\[ S_{21}^{e} = \frac{S_{21i}^{th}}{\delta_{At}} \]  \hspace{1cm} (5.16)

since the experimental and theoretical values should be equal, set \( S_{21}^{expMS} = S_{21}^{e} \). Next inserting in the experimental and theoretical solution into the following equation:

\[ \text{RelativeError} = \frac{\text{Experimental} - \text{Theoretical}}{\text{Theoretical}} \]  \hspace{1cm} (5.17)

This then gives the total error associated with \( S_{21}^{e} \) which directly relates to \( \epsilon_{xx} \).

\[ \epsilon_{xx} \text{RelativeError} = \delta_{At} \left[ \left( \cos^2(\phi) + \frac{S_{21}^{yyMS}}{S_{21}^{e}} \sin^2(\phi) \right) \delta_{At} e^{-jK_{0}(\delta t_{d} + \delta d_{u})} - 1 \right] \]  \hspace{1cm} (5.18)

Following a similar method, the error associated with the \( \epsilon_{yy} \) material parameters can be found to be equal to the following:

\[ \epsilon_{yy} \text{RelativeError} = \delta_{At} \left[ \left( \cos^2(\phi) + \frac{S_{21}^{xxMS}}{S_{21}^{e}} \sin^2(\phi) \right) \delta_{At} e^{-jK_{0}(\delta t_{d} + \delta d_{u})} - 1 \right] \]  \hspace{1cm} (5.19)

5.2.1.2 \( \hat{z} \) Error

\( \hat{z} \) follows a similar method used when calculating \( \hat{x} \) and \( \hat{y} \), but the error associated with \( \theta \) needs to be included. Taking Equation 2.56 with error terms gives the following:

\[ S_{21}^{MS} = \frac{S_{21}^{MS}}{S_{21}^{e}} e^{-jK_{0}(\delta d_{i}(\cos t_{d} + \delta d_{u}) + \delta d_{u} + \delta d_{u}) \cos(\theta_{d} + \delta \theta_{d})} \]  \hspace{1cm} (5.20)
To simplify these calculations, the error term associated with $\theta$ will be simplified as follows:

$$cos(\theta + \delta) = cos(\theta)cos(\delta) - sin(\theta)sin(\delta)$$ (5.21)

$$sin(\theta + \delta) = sin(\theta)cos(\delta) + cos(\theta)sin(\delta)$$ (5.22)

since $\delta$ is assumed to be small, $cos(\delta) \sim 1$, $sin(\delta) \sim \delta$

$$cos(\theta + \delta) \sim cos(\theta) - \delta sin(\theta)$$ (5.23)

$$sin(\theta + \delta) \sim sin(\theta) + \delta cos(\theta)$$ (5.24)

$$cos(\theta_i + \delta_{\theta_i}) \sim cos(\theta_i) - \delta sin(\theta_i) = cos(\theta_i) - \delta_{\theta_i}$$ (5.25)

$$sin(\theta_i + \delta_{\theta_i}) \sim sin(\theta_i) + \delta cos(\theta_i) = sin(\theta_i) + \delta_{\theta_i}$$ (5.26)

Using this relationship, the Equation 5.20 can be simplified to the following:

$$S_{x_{21}}^{x_{21}MS} = \frac{S_{x_{21}^{x_{21}MS}}}{S_{x_{21}^{x_{21}MT}}} e^{-jK_0((d_i + \delta_{d_i})(cos(\theta_i) - \delta_{\theta_i}) + (d_u + \delta_{d_u})(cos(\theta_u) - \delta_{\theta_u}))}$$ (5.27)

Separating out the error term, the previous equation can be simplified to

$$S_{x_{21}}^{x_{21}MS} = S_{x_{21}}^{x_{21}MS} e^{-jK_0(\delta_{d_i}(cos(\theta_i) - \delta_{\theta_i}) - d_i \delta_{\theta_i} + \delta_{d_u}(cos(\theta_u) - \delta_{\theta_u}) - d_u \delta_{\theta_u})} = S_{x_{21}}^{x_{21}MS} \delta_z$$ (5.28)

Next, using Equation 3.137a, the error caused by not aligning the axis of the sample with the antenna.

$$S_{x_{21}}^{x_{21}MS} = ([cos^2(\theta + \delta_0)]S_{x_{21}}^{x_{21}MS}cos^2(\delta_\phi) + S_{uv}^{uv} sin^2(\delta_\phi)) + sin^2(\theta + \delta_0)S_{zz}^{zz})\delta_z$$ (5.29)

To separate the angles inside the trig terms, for cosine, use Equation 5.25 and for sine, use Equation 5.26.
Setting $\delta_{zt}$ to

$$S_{21}^{MSe} = ([\cos^2(\theta) - 2 \cos(\theta)\delta_s + \delta^2_s]S^{xx} \cos^2(\delta_\phi) + S^{yy} \sin^2(\delta_\phi)] + (\sin^2(\theta) + 2 \sin(\theta)\delta_c + \delta^2_c)S^{zz})\delta_z$$

(5.30)

Next the theoretical error needs to be calculated following a similar process used previously with the additional rotation of the sample. $P$, from Equation 5.5, changes to:

$$P_e = e^{-jK_0(d_u + \delta_d)\cos(\theta + \delta_\theta)} = P e^{-jK_0(\delta_d \cos(\theta) - \delta_\theta (d_u + \delta_u)}$$

(5.33)

Using Equation 5.33, Equation 5.7 becomes

$$\delta S_{21z} = \frac{(1 - R^2 P^2) e^{-jK_0(\delta_d \cos(\theta) - \delta_\theta (d_u + \delta_u)}}{1 - P^2 R^2 e^{-2jK_0(\delta_d \cos(\theta) - \delta_\theta (d_u + \delta_u)}}$$

(5.34)

and Equation 5.10 becomes

$$\delta S_{11z} = \frac{(1 - P^2 e^{-2jK_0(\delta_d \cos(\theta) - \delta_\theta (d_u + \delta_u)}) (1 - P^2 R^2)}{1 - P^2 R^2 e^{-2jK_0(\delta_d \cos(\theta) - \delta_\theta (d_u + \delta_u)}}$$

(5.35)

$\delta S_{21z}$ is the error associated with $S_{21}$ when rotated about the Y axis, and $\delta S_{11z}$ is the error associated with $S_{11}$ when rotated around the Y axis.

Following the same calculations as earlier, $\delta A_4$ from Equation 5.15 is then modified to equal $\delta_{Atz}$.

$$\delta_{A_{tz}} = \frac{A_{111} A_{11b} + A_{12i} A_{21b} \delta_{11z} \delta_{11bz}}{A_{111} \delta_{21z} \delta_{21bz}}$$

(5.36)

Lastly, using Equation 5.17 the total relative error can be found.
\[
\epsilon_z \text{RelativeError} = \frac{(S_{xx}^2 + \delta_{xt})\delta_z - S_{21t}^2}{S_{21t}^2 \delta_{Atz}}
\] (5.37)

\[
\epsilon_z \text{RelativeError} = \delta_{Atz} \left( S_{xx}^2 + \frac{\delta_{xt}}{S_{21t}^2} \right) \delta_z - 1
\] (5.38)

Since the Z axis calculation is dependent on the value found for the X axis through Equation 3.23, setting relative permeability equal to 1 creates the following:

\[
K^\parallel = \pm \sqrt{\frac{\omega^2 \epsilon_{xx}^e \epsilon_{zz}}{\cos^2(\theta) \epsilon_{zz} + \sin^2(\theta) \epsilon_{xx}^e}}
\] (5.39)

with \(\epsilon_{xx}^e\) representing the \(\hat{x}\) permittivity found using the error in Section 5.2.1.1. As \(\theta\) gets smaller, the variable \(\epsilon_z\) approaches 1 making the unknown \(\epsilon_z\) a much smaller portion of the fraction. This causes any errors inside of \(\epsilon_{xx}\) to have a disproportionately large effect of the value of \(K^\parallel\) changing any small error in \(\epsilon_{zz}\) to be much larger in the solution for \(\epsilon_{zz}\).

### 5.2.2 Biaxial Equivalency Method Error

For the Equivalency method error, there still exists an error associated with the sample thickness, but no angle errors as there is no rotation around the Y axis and the Z axis is set as unknown, removing that assumption. There also exists the error associated with approximating certain S parameters, giving rise to the following error terms:

\[
d^e_u = d_u + \delta_{du}
\] (5.40a)

\[
d^e_i = d_i + \delta_{di}
\] (5.40b)

\[
S_{xx}^{Be} = S_{12}^{Be} \delta_{y_2}^{12}
\] (5.40c)

\[
S_{yy}^{Be} = S_{21}^{Be} \delta_{y_2}^{21}
\] (5.40d)

\[
S_{yy}^{Be} = S_{12}^{Be} \delta_{y_2}^{12}
\] (5.40e)

\[
S_{xx}^{Be} = S_{21}^{Be} \delta_{y_2}^{21}
\] (5.40f)

Where \(\delta^{21}\) represents equating the X and Y orientations for a wave propagating from antenna 1 to antenna 2 and \(\delta^{12}\) representing equating the X and Y orientations for a wave propagating from antenna 2 and antenna 2. Starting with Equations 3.118, then substituting in the above error.
terms gives the following:

\[
S_{21\phi}^x S_{12\phi}^{y'} S_{21\phi}^{x'} S_{12\phi}^{y'} = \frac{S_{xy\phi}^{yMS} S_{xy\phi}^{mMS}}{S_{xy\phi}^{yB} S_{xy\phi}^{yA} S_{xy\phi}^{yB} S_{xy\phi}^{yA} (xy)^2 y x} = \frac{S_{xy\phi}^{yMS} S_{xy\phi}^{mMS}}{S_{xy\phi}^{yB} S_{xy\phi}^{yA}} e^{-2jK_0(d + \delta_d)}
\]

(5.41a)

\[
S_{21\phi}^y S_{12\phi}^{x'} S_{21\phi}^{y'} S_{12\phi}^{x'} = \frac{S_{yx\phi}^{yMS} S_{yx\phi}^{mMS}}{S_{yx\phi}^{yB} S_{yx\phi}^{yA} S_{yx\phi}^{yB} S_{yx\phi}^{yA} (yx)^2 y x} = \frac{S_{yx\phi}^{yMS} S_{yx\phi}^{mMS}}{S_{yx\phi}^{yB} S_{yx\phi}^{yA}} e^{-2jK_0(d + \delta_d)}
\]

(5.41b)

Using Equation 3.120 and Equation 3.121, then isolating the error terms gives:

\[
S_{21\phi}^{x' x'} e^{-jK_0(\delta_{d_1} + \delta_{d_2})} = \frac{S_{21\phi}^{x' x'}}{\sqrt{\delta_{21\phi}^y \delta_{12}^y}}
\]

(5.42a)

\[
S_{21\phi}^{y' y'} e^{-jK_0(\delta_{d_1} + \delta_{d_2})} = \frac{S_{21\phi}^{y' y'}}{\sqrt{\delta_{21\phi}^x \delta_{12}^x}}
\]

(5.42b)

where \( S_{21\phi}^{x' y'} e^{-jK_0(\delta_{d_1} + \delta_{d_2})} \) is the cross polarization with error equal to the true value multiplied by the error factor.

For \( S_{21\phi}^{x' x'} \) and \( S_{21\phi}^{y' y'} \), the only error that is induced is from the sample thickness leading to the following:

\[
S_{21\phi}^{x' x'} = \frac{S_{21\phi}^{x' x'}}{e^{-jK_0(\delta_{d_1} + \delta_{d_2})}}
\]

(5.43a)

\[
S_{21\phi}^{y' y'} = \frac{S_{21\phi}^{y' y'}}{e^{-jK_0(\delta_{d_1} + \delta_{d_2})}}
\]

(5.43b)

inserting these into Equation 3.100a and Equation 3.100d

\[
S^{xxe} = \cos^2(\phi) S_{21\phi}^{x' x'} - \sin \phi \cos \phi \left( \frac{S_{21\phi}^{y' y'}}{\sqrt{\delta_{21\phi}^y \delta_{12}^y}} + \frac{S_{21\phi}^{y' y'}}{\sqrt{\delta_{21\phi}^y \delta_{12}^y}} \right) + \sin^2(\phi) S_{21\phi}^{y' y'} e^{-jK_0(\delta_{d_1} + \delta_{d_2})}
\]

(5.44)

\[
S^{yye} = \sin^2(\phi) S_{21\phi}^{x' x'} + \sin \phi \cos \phi \left( \frac{S_{21\phi}^{y' y'}}{\sqrt{\delta_{21\phi}^y \delta_{12}^y}} + \frac{S_{21\phi}^{y' y'}}{\sqrt{\delta_{21\phi}^y \delta_{12}^y}} \right) + \cos^2(\phi) S_{21\phi}^{y' y'} e^{-jK_0(\delta_{d_1} + \delta_{d_2})}
\]

(5.45)
and setting

\[
\delta_{xx} = \frac{S_{x'}y'}{\sqrt{\delta_{xx}^2 + \delta_{yy}^2}} + \frac{S_{x'y'}}{\sqrt{\delta_{xx}^2 + \delta_{yy}^2}} - S_{x'y'} - S_{x'y'} 
\]

\[
\delta_{yy} = \frac{S_{x'y'}}{\sqrt{\delta_{xx}^2 + \delta_{yy}^2}} + \frac{S_{x'y'}}{\sqrt{\delta_{xx}^2 + \delta_{yy}^2}} - S_{x'y'} - S_{x'y'} 
\]

Equation 5.44 and Equation 5.45 can then be simplified to

\[
S_{xx} = \left( S_{xx} - \frac{1}{2} \sin(2\phi) \delta_{xx} \right) e^{-jK_a(\delta_{dx} + \delta_{dy})} 
\]

\[
S_{yy} = \left( S_{yy} + \frac{1}{2} \sin(2\phi) \delta_{yy} \right) e^{-jK_a(\delta_{dx} + \delta_{dy})} 
\]

Taking the difference between this and the theoretical values from Equation 5.16 gives:

\[
0 = \left( S_{xx} - \frac{1}{2} \sin(2\phi) \delta_{xx} \right) e^{-jK_a(\delta_{dx} + \delta_{dy})} - \frac{S_{xx}^{21t}}{\delta_{Atx}} 
\]

\[
0 = \left( S_{yy} + \frac{1}{2} \sin(2\phi) \delta_{yy} \right) e^{-jK_a(\delta_{dx} + \delta_{dy})} - \frac{S_{yy}^{21t}}{\delta_{Aty}} 
\]

Lastly, using Equation 5.17, the error for the \( \hat{x} \) and \( \hat{y} \) material parameters can be found.

\[
\epsilon_{xx} Relative Error = \left( 1 - \frac{\sin(2\phi) \delta_{xx}}{2S_{xx}} \right) \delta_{Atx} e^{-jK_a(\delta_{dx} + \delta_{dy})} - 1 
\]

\[
\epsilon_{yy} Relative Error = \left( 1 + \frac{\sin(2\phi) \delta_{yy}}{2S_{yy}} \right) \delta_{Aty} e^{-jK_a(\delta_{dx} + \delta_{dy})} - 1 
\]

This represents the error in the S terms when solving for the material parameters using the Equivalency method.

5.2.3 Second Sample Method Error

The second sample method has similar errors as the previous one except that instead of induced errors from approximating an equivalence, errors from using a second sample for
calibration are introduced giving rise to the following:

\[
d_u^e = d_u + \delta d_u \quad (5.50a)
\]
\[
d_i^e = d_i + \delta d_i \quad (5.50b)
\]
\[
d_2^e = d_2 + \delta d_2 \quad (5.50c)
\]
\[
\phi_0^e = \delta \phi_0 \quad (5.50d)
\]
\[
\phi_2^e = \phi + \delta \phi_2 \quad (5.50e)
\]

where \(\delta d_2\) is the error in the thickness of sample two, \(\delta \phi_0\) is the error in the second sample’s unrotated angle measurement, and \(\delta \phi_2\) is the error in the second sample’s rotated angle measurement.

The error induced by the unrotated second sample is the same as what was found in Section 5.2.1.1 using Equation 5.4.

\[
S_{xx}^{e21} = (S_{xx}^{21} \cos^2(\delta \phi_0) + S_{yy}^{21} \sin^2(\delta \phi_0)) e^{-jK_0(\delta d_1 + \delta d_2)} \quad (5.51a)
\]
\[
S_{yy}^{e21} = (S_{yy}^{21} \cos^2(\delta \phi_0) + S_{xx}^{21} \sin^2(\delta \phi_0)) e^{-jK_0(\delta d_1 + \delta d_2)} \quad (5.51b)
\]

setting

\[
\delta_{xx0} = \cos^2(\delta \phi_0) + \frac{S_{yy}^{21}}{S_{xx}^{21}} \sin^2(\delta \phi_0) \quad (5.52a)
\]
\[
\delta_{yy0} = \cos^2(\delta \phi_0) + \frac{S_{xx}^{21}}{S_{yy}^{21}} \sin^2(\delta \phi_0) \quad (5.52b)
\]

the following relationship is obtained

\[
S_{xx}^{e21} = S_{xx}^{21} \delta_{xx0} e^{-jK_0(\delta d_1 + \delta d_2)} \quad (5.53a)
\]
\[
S_{yy}^{e21} = S_{yy}^{21} \delta_{yy0} e^{-jK_0(\delta d_1 + \delta d_2)} \quad (5.53b)
\]

The rotated second sample \(S\) value can be calculated using Equation 3.97.
\[
S^{xy2e}_{21} = S^{yx2e}_{21} = \sin(\phi_2 + \delta_{\phi2}) \cos(\phi_2 + \delta_{\phi2})(S^{xy2e}_{21} - S^{yx2e}_{21}) \quad (5.54)
\]

\[
S^{xy2e}_{21} = S^{yx2e}_{21} = (\sin(\phi_2) + \delta_{c\phi2})(\cos(\phi_2) - \delta_{s\phi2})(S^{xy2e}_{21} \delta_{yy0} - S^{yx2e}_{21} \delta_{xx0})e^{-jK_0(\delta_{d\phi} + \delta_{d\phi})} \quad (5.55)
\]

Setting

\[
\delta_{xy2} = \left( \frac{1}{S^{yy2e}_{21} - S^{xx2e}_{21}} + \cos(\phi)\delta_{c\phi2} - \sin(\phi)\delta_{s\phi2} - \delta_{c\phi2}\delta_{c\phi2}\right) \left( S^{yy2e}_{21} \delta_{yy0} - S^{xx2e}_{21} \delta_{xx0} \right) \quad (5.56)
\]

results in

\[
S^{xy2e}_{21} = S^{yx2e}_{21} = S^{xy2e}_{21} \delta_{xy2} e^{-jK_0(\delta_{d\phi} + \delta_{d\phi})} = S^{yx2e}_{21} \delta_{xy2} e^{-jK_0(\delta_{d\phi} + \delta_{d\phi})} \quad (5.57)
\]

The same method can be applied to \( S_{12} \) resulting in the following:

\[
S^{xy2e}_{12} = S^{yx2e}_{12} = S^{xy2e}_{12} \delta_{xy2} e^{-jK_0(\delta_{d\phi} + \delta_{d\phi})} = S^{yx2e}_{12} \delta_{xy2} e^{-jK_0(\delta_{d\phi} + \delta_{d\phi})} \quad (5.58)
\]

Equation 5.58 can then be incorporated into Equation 3.118, and isolate \( S_{21} \).

\[
S^{x'e'e}_2 e = S^{y'x'e}_2 e^{-jK_0(\delta_{d\phi} + \delta_{d\phi})} \quad S^{y'e'e}_2 e = S^{y'x'e}_2 e^{-jK_0(\delta_{d\phi} + \delta_{d\phi})} \quad (5.59)
\]

substituting these into Equation 3.100a and Equation 3.100d to obtain

\[
S^{xx'e}_2 = (\cos^2(\phi)S^{x'x'} + \sin \phi \cos \phi(S^{y'x'} \delta_{xy2} + S^{x'y'} \delta_{xy2}) + \sin^2(\phi)S^{y'y'})e^{-jK_0(\delta_{d\phi} + \delta_{d\phi})} \quad (5.60)
\]

\[
S^{yx'e}_2 = (\sin^2(\phi)S^{x'x'} + \sin \phi \cos \phi(S^{y'x'} \delta_{xy2} + S^{x'y'} \delta_{xy2}) + \cos^2(\phi)S^{y'y'})e^{-jK_0(\delta_{d\phi} + \delta_{d\phi})} \quad (5.61)
\]

setting

\[
\delta_2 = (S^{x'y'} + S^{y'x'})(\delta_{xy2} - 1) \quad (5.62)
\]
leads to

\[ S_{21}^{xx} = (S_{21}^{xx} + \delta_2)e^{-jK_0(\delta_{du} + \delta_{di})} \]  \hspace{1cm} (5.63a)
\[ S_{21}^{yy} = (S_{21}^{yy} + \delta_2)e^{-jK_0(\delta_{du} + \delta_{di})} \]  \hspace{1cm} (5.63b)

Taking the difference between this and the theoretical values from Equation 5.16 gives

\[ 0 = (S_{xx}^{xx} + \delta_2)e^{-jK_0(\delta_{du} + \delta_{di})} - \frac{S_{21}^{xx}}{\delta_{Ax}} \]  \hspace{1cm} (5.64a)
\[ 0 = (S_{yy}^{yy} + \delta_2)e^{-jK_0(\delta_{du} + \delta_{di})} - \frac{S_{21}^{yy}}{\delta_{Ay}} \]  \hspace{1cm} (5.64b)

The error can then be calculated for \( \hat{x} \) and \( \hat{y} \) S parameter using Equation 5.17.

\[ \epsilon_{xx} \text{Relative Error} = \left( 1 + \frac{\delta_2}{S_{xx}^{xx}} \right) \delta_{Ax}e^{-jK_0(\delta_{du} + \delta_{di})} - 1 \]  \hspace{1cm} (5.65a)
\[ \epsilon_{yy} \text{Relative Error} = \left( 1 + \frac{\delta_2}{S_{yy}^{yy}} \right) \delta_{Ay}e^{-jK_0(\delta_{du} + \delta_{di})} - 1 \]  \hspace{1cm} (5.65b)

5.2.4 Oblique Incidence with Z Axis Rotation Error

The measurement errors for extracting the \( \hat{z} \) material parameters using the second sample method with a known rotation around the Y axis, and an unknown rotation around the Z axis. This methodology possesses the same error terms as the second sample method but now also includes the error in the measured angle \( \theta \) as shown in Equation 5.66a.

\[ d_u^c = d_u + \delta_{du} \]  \hspace{1cm} (5.66a)
\[ d_i^c = d_i + \delta_{di} \]  \hspace{1cm} (5.66b)
\[ d_2^c = d_2 + \delta_{d2} \]  \hspace{1cm} (5.66c)
\[ \phi_0^c = \delta_{\phi 0} \]  \hspace{1cm} (5.66d)
\[ \phi_2^c = \phi + \delta_{\phi 2} \]  \hspace{1cm} (5.66e)
\[ \theta^c = \theta + \delta_{\theta} \]  \hspace{1cm} (5.66f)
Taking Equation 3.142 the error associated with the V calculations is as follows

\[ V^e = (d_2 + \delta_{d_2})(\sec(\theta + \delta_\theta) - 1) \]  

(5.67)

or

\[ V^e = (d_2 + \delta_{d_2})(\sec(\theta) + \delta_{sec\theta} - 1) \]  

(5.68)

which can have the error term separated to give

\[ V^e = V + d_2(\delta_{sec\theta}) + \delta_d(\sec(\theta) + \delta_{sec\theta} - 1) = V + \delta_V \]  

(5.69)

substituting \( V^e \) into Equation 3.144 along with the error terms from Equation 5.57, Equation 5.58, and Equation 5.28, then isolating \( S_{21} \) results in the following:

\[ S_{21\theta\phi}^{x'y'xe} = S_{21\theta\phi}^{x'y'd_{xy}2\delta_z - jK_0(\delta_{u_1} + \delta_{u_2} + \delta_V)} = S_{21\theta\phi}^{x'y'd_{xyz} \delta_z} \]  

(5.70a)

\[ S_{21\theta\phi}^{y'x'e} = S_{21\theta\phi}^{y'x'd_{xy}2\delta_z - jK_0(\delta_{u_1} + \delta_{u_2} + \delta_V)} = S_{21\theta\phi}^{y'x'd_{xyz} \delta_z} \]  

(5.70b)

The error associated with the xx and yy S parameters are calibrated the same way as Equation 5.28 shown below.

\[ S_{21\theta\phi}^{x'x'e} = S_{21\theta\phi}^{x'x'\delta_z} \]  

(5.71a)

\[ S_{21\theta\phi}^{y'y'e} = S_{21\theta\phi}^{y'y'\delta_z} \]  

(5.71b)

Using these terms, \( S_{21}^{x'xe} \) and \( S_{21}^{y'ye} \) are isolated using Equation 3.100a and Equation 3.100d.

\[ S_{21}^{x'xe} = (\cos^2(\phi)S_{21\theta\phi}^{x'x'}) - \sin \phi \cos \phi (S_{21\theta\phi}^{y'y'} + S_{21\theta\phi}^{x'y'})\delta_{xyz} + \sin^2(\phi)S_{21\theta\phi}^{y'y'} \delta_z \]  

(5.72)

\[ S_{21\theta\phi}^{y'ye} = (\sin^2(\phi)S_{21\theta\phi}^{y'y'}) + \sin \phi \cos \phi (S_{21\theta\phi}^{y'y'} + S_{21\theta\phi}^{x'y'})\delta_{xyz} + \cos^2(\phi)S_{21\theta\phi}^{y'y'} \delta_z \]  

(5.73)
defining
\[ \delta_{xyz} = (S_{210}^\phi + S_{210}^\phi)(\delta_{x_{yz}} - 1) \] (5.74)

Equation 5.72 and Equation 5.73 can be simplified to

\[ S_{21}^{xx} = (S_{21}^{xx} - \delta_{xyz})\delta_z \] (5.75a)
\[ S_{21}^{yy} = (S_{21}^{yy} + \delta_{xyz})\delta_z \] (5.75b)

Using Equation 3.97, the rotated material parameters without cross polarization can be found as

\[ S_{21}^{x'x'}^\phi = (\cos^2(\phi)((S_{21}^{xx} - \delta_{xyz})\delta_z) + \sin^2(\phi)((S_{21}^{yy} + \delta_{xyz})\delta_z) \] (5.76)

Separating out the error terms gives

\[ S_{21}^{x'x'}^\phi = [S_{21}^{x'x'} + (1 - 2 \cos^2(\theta))\delta_{xyz}]\delta_z \] (5.77)

Since the theoretical error for this S parameter depends on both the xx and yy S parameters, Equation 5.36 is incorporated into Equation 3.97 to give the theoretical rotated value.

\[ S_{21}^{x'x'}^\phi = \cos^2(\phi)\frac{S_{21}^{xx}}{\delta_{Atz}} + \sin^2(\theta)\frac{S_{21}^{yy}}{\delta_{Atz}} \] (5.78)

separating out the error terms gives

\[ S_{21}^{x'x'}^\phi = \frac{S_{21}^{x'x'}^\phi}{\delta_{Atz}} \] (5.79)

Lastly, using Equation 5.17, the error associated with this method can be found.

\[ \varepsilon_{zzRelativeError} = \left[ 1 + \frac{(1 - 2 \cos^2(\theta)\delta_{xyz})\delta_{xyz}}{S_{21}^{x'x'}}\right] \delta_{Atz} - 1 \] (5.80)

As with the previous \( \hat{z} \) material parameter calculation, this measurement is also dependent on the values found previously for the xx and yy material parameters. However, due to the rotation, Equation 3.23 must use \( \varepsilon_{x'x'} \) which can be calculated using Equation 3.148a giving it the errors that were induced in both the \( \varepsilon_{xx} \) and \( \varepsilon_{yy} \). \( \varepsilon_{x'x'} \) is then magnified by the \( \sin^2(\theta) \) multiplication in the denominator of the equation.
\[ K^\parallel = \pm \sqrt{\frac{\omega^2 \epsilon_{x'x}^\epsilon \epsilon_{zz}}{\cos^2(\theta) \epsilon_{zz} + \sin^2(\theta) \epsilon_{x'x}'}} } \] (5.81)

5.3 Monte Carlo Error Analysis

Another method to analyzing the error in a data set is the Monte Carlo method. To analyze the data for this thesis, each measurement was taken with a \( \pm \delta \) representing the possible error in that measurement shown in figure 5.1.

\[
\begin{array}{c}
-\delta X \\
\chi \\
+\delta X \\
\end{array}
\]

Figure 5.1. Uniform Distribution

A random number generator was then used to produce 1000 random samples with values that reside within this range using a uniform distribution set. The values for each method are as follows:

**Axes Aligned with Antenna**

Thickness of isotropic sample \( (d_i) \), thickness of biaxial Sample \( (d_u) \), internal angle of isotropic sample \( (\theta_i) \), and internal angle of biaxial sample \( (\theta_u) \)

\[ d_i = \overline{d_i} \pm \delta d_i \] (5.82)

\[ d_u = \overline{d_u} \pm \delta d_u \] (5.83)

\[ \theta_i = \overline{\theta_i} \pm \delta \theta \] (5.84)

\[ \theta_u = \overline{\theta_u} \pm \delta \theta \] (5.85)

**Equivalency Method**

Thickness of isotropic sample \( (d_i) \) and thickness of biaxial Sample \( (d_u) \).

\[ d_i = \overline{d_i} \pm \delta d_i \] (5.86)
\[ d_u = \bar{d}_u \pm \delta d_u \] (5.87)

**Second Sample Method X and Y axes**

Thickness of isotropic sample \((d_i)\), thickness of biaxial Sample \((d_u)\), thickness of second sample and angle of rotated second sample \((\phi_2)\).

\[ d_i = \bar{d}_i \pm \delta d_i \] (5.88)

\[ d_u = \bar{d}_u \pm \delta d_u \] (5.89)

\[ d_2 = \bar{d}_2 \pm \delta d_2 \] (5.90)

\[ \phi_2 = \bar{\phi}_2 \pm \delta \phi \] (5.91)

**second Sample Method all 3 axes**

Thickness of isotropic sample \((d_i)\), thickness of biaxial Sample \((d_u)\), thickness of second sample, internal angle of isotropic sample \((\theta_i)\), internal angle of biaxial sample \((\theta_u)\) and angle of rotated second sample \((\phi_2)\).

\[ d_i = \bar{d}_i \pm \delta d_i \] (5.92)

\[ d_u = \bar{d}_u \pm \delta d_2 \] (5.93)

\[ d_2 = \bar{d}_2 \pm \delta d_2 \] (5.94)

\[ \theta_i = \bar{\theta}_i \pm \delta \theta \] (5.95)

\[ \theta_u = \bar{\theta}_u \pm \delta \theta \] (5.96)

5-16
\[ \phi_2 = \overline{\theta}_2 \pm \delta \phi \]  

(5.97)

Running these 1000 data sets through the equations to calculate the permittivity of the sample, a value is obtained for each iteration. Next the mean of these permittivities were calculated for each frequency.

\[ \overline{\epsilon}_r = \frac{1}{N} \sum_{i=1}^{N} \epsilon_{r,i} \]  

(5.98)

next, the standard deviation was calculated.

\[ \delta \epsilon_r = \sqrt{\frac{\sum_{i=1}^{N} (\epsilon_{r,i} - \overline{\epsilon}_r)^2}{N}} \]  

(5.99)

This standard deviation calculated for each frequency was then used to find the expected error in permittivity to generate the error bars found in Section 6..

### 5.4 Error Overview

For calculating the $\hat{z}$ material parameters, the results inherently possess a larger amount of error when compared to the $\hat{x}$ and $\hat{y}$ material parameters. This error is due to the $\hat{x}$ and $\hat{y}$ calculations errors carrying over into the $\hat{z}$ errors and magnified on top of any inherent errors from $\hat{z}$.

From experimental data, one of the variables that most affect the results is the thickness of the sample being measured. A way to improve the measurement for this is to obtain a device to measure the thickness of the sample in its center instead of around the edges, as that is where the highest concentration of EM waves will pass through. An average thickness from multiple measurements at different locations around the center would also help to mitigate this issue. Another way to increase the accuracy is to make sure that all measured angles are accurately representing. To help improve this, a digital angle measuring device could be used. The angle accuracy for the Y axis rotation did not possess as much of an effect on the resulting solutions when compared to sample thickness or alignment of the sample axis with the antenna when that assumption is made but can also be improved using such a device. Lastly removing as much interference from the room as possible by minimizing any EM waves that are not involved in the experiment to help reduce the negative effects of the FBS open system.
6. Results

Following the methodology detailed in Section 4 and the error calculations in Section 5, the resulting solutions are discussed for each method derived in this thesis.

6.1 Biaxial Sample with Oblique Incidence

![Graph showing Biaxial Sample Y Axis Rotation Permittivity vs Frequency]

Figure 6.1. Biaxial Sample Y Axis Rotation Sample 1 Complex Results
Figure 6.2. Biaxial Sample Y Axis Rotation Sample 1 Real Results

Figure 6.3. Biaxial Sample Y Axis Rotation Sample 1 Averaged
From Figure 6.1, the imaginary portion of the $\hat{x}$ and $\hat{y}$ permittivity are both zero as expected for the sample. The $\hat{z}$ imaginary portion averages around zero but has larger fluctuations in the values. This matches what would be expected from the error analysis in Section 5., since the fluctuations in the $\hat{x}$ calculations are magnified in the $\hat{z}$ calculations. The same pattern be seen in the real portion of $\hat{z}$ (Figure 6.2). The real portion of $\epsilon_{zz}$ stays almost constant when averaged as shown in Figure 6.3 demonstrating that though the calculation gives large fluctuations, the value that it fluctuations around stays fairly consistent. The results for all three material parameters start to get larger fluctuations as the frequency continues to increase. This is the result the EM waves passing through the sample becoming less of an ideal plane wave.

![Figure 6.4. Biaxial Sample Y Axis Rotation Sample 2 Complex Results](image)

Figure 6.4. Biaxial Sample Y Axis Rotation Sample 2 Complex Results
Figure 6.5. Biaxial Sample Y Axis Rotation Sample 2 Real Results

Figure 6.6. Biaxial Sample Y Axis Rotation Sample 2 Real Results
For sample 2 shown in Figure 6.4, the same general observations for sample 1 can still be seen. One noticeable difference is that the real portion of \( \hat{\varepsilon} \) permittivity does not fluctuate around a value as consistently and the imaginary portion no longer remains centered on zero. There are a couple possible explanations for this discrepancy. One is that the sample is not aligned with the axis of the antenna. When looking at Equation 3.137 and the error calculations in Section 5., the error caused by axial misalignment when a second axis of rotation is induced are larger then when only one axis rotation is done and the effects on \( \varepsilon_x \) and \( \varepsilon_y \) would mostly only cause the two axes to slightly shift the other. The induced cross polarization from a second axial rotation could also explain the non-zero imaginary portion of the z axis permittivity. Another reason to think this is because when using the equivalency method and the second sample method a 7 degree axial misalignment was found when the sample was lined up with the antennas. This would also leads to slightly different material parameters for both. This can be seen when comparing Figures 6.7, 6.11, and 6.14. Since 7 degrees is larger than the margin of error for the axial alignment of the sample, the internal axis might not align with the sides of the sample causing an increase in axial misalignment or an inherent cross polarization in the sample exists. It can also be seen in Figure 6.17 that when the angle of rotation around the Z axis is no longer assumed, the calculations straighten out more and the imaginary portion centers on zero.
A similar material was measured using a waveguide rectangular to waveguide square (WRWS) [11] system shown in Figure 6.8/6.9 and using a waveguide flange shown in Figure 6.10.

Figure 6.8. Uniaxial Cube WRWS Measurements Low Contrast [4]
Comparing the results from Figure 6.8, one can see that the $\epsilon_r(01/05)$, that have the same design as the $\hat{z}$ permittivity of both samples, possesses similar values as the results from Figure 6.1 and Figure 6.4 shows. The Z axis for sample 1 matches almost exactly what was found using the WRWS while the values found for sample 2 match those obtained using a waveguide-flange.

Looking at the $\hat{x}$ and $\hat{y}$ permittivity for samples 1 and 2, the values obtained for the axis match the values shown in Figure 6.8. This comparison shows that the experiment obtain reasonable results.
6.2 Rotated Biaxial Sample Using Equivalency Method

When testing the equivalency method, sample 2 was measured at three different angles, 0 degrees (Figure 6.11), 30 degrees (Figure 6.12) and 45 degrees (Figure 6.13).

Looking at Figure 6.11, the sample possesses consistent results across the frequency range and at values about what had been found previously.

![Figure 6.11. Equivalency Method With Sample Rotated 0 Degrees](image)

As the sample is rotated, fluctuations in the data begin to appear, shown in Figure 6.12. These fluctuations are believed to correlate to the internal resonance frequency, or varying internal impedance inside different parts of the antenna. Since the equivalency method equates different axes of the antenna, these fluctuations are longer removed when calibrated, causing them to appear in the final solutions. The values remain around what is expected but is not desirable if very precise solutions are required.
Figure 6.12. Equivalency Method with Sample Rotated 30 Degrees

Figure 6.13. Equivalency Method with Sample Rotated 45 Degrees
Similar results are shown in Figure 6.13 which possess the largest possible error due to the rotation that is possible (45 degrees) and the solution remains around the expected values for the permittivity of the sample.

6.3 Rotated Biaxial Sample Using Second Sample Method

For the second sample method, three measurements were again taken at 0 degrees (Figure 6.14), 30 degrees (Figure 6.15), and 45 degrees (Figure 6.16).

Figure 6.14 possesses a straight line as desired. It also does not possess the fluctuations found in the previous methodology. The downside of this method is that it requires a larger number of measurements and a second biaxial sample that needs to be orientated very carefully for the calibration to work.

![Figure 6.14. Second Sample Method with Sample Rotated 0 Degrees](image)

In Figure 6.15 at 30 degrees and Figure 6.16 at 45 degrees, the slight up-shift found in the previous method still persists giving further evidence that this up-shift is due to the sample possessing inherent cross polarization.
Figure 6.15. Second Sample Method with Sample Rotated 30 Degrees

Figure 6.16. Second Sample Method with Sample Rotated 45 Degrees
6.4 Rotated Biaxial Sample Around Z and Y axis using 2nd Sample

As can be seen in the equivalency method graphs, the solutions possess a very large amount of fluctuations across frequencies, making it undesirable when doing \( \hat{z} \) calculations. Since any errors will be magnified when finding the \( \hat{z} \) permittivity, the second sample method needs to be used.

When comparing results, the \( \hat{z} \) material parameters in Figure 6.17 possess more fluctuations than found in Figure 6.4 but does retain a more consistent medium across frequencies with the assumption of axis alignment removed as can be seen in Figure 6.19. It can also be seen that the imaginary portion of the \( \hat{z} \) no longer increases and instead stays centered on zero as desired. The real portion also resides in the same area as was seen in Figure 6.17, showing that this method obtains accurate results.

![Biaxial Focus Beam System Permittivity vs Frequency Z & Y Rotation](image)

Figure 6.17. Second Sample Method for All 3 Material Parameters Complex Results
Figure 6.18. Second Sample Method for All 3 Material Parameters Real Results

Figure 6.19. Second Sample Method for All 3 Material Parameters Averaged
7. Conclusion and Recommendations

This thesis investigates the method to extract the diagonal material parameters of a biaxial material utilizing a FBS. The methods were derived and tested with the results discussed in Section 6. It was shown that the methodologies presented were able to get accurate solutions for all three material parameters of a sample. This was achieved for the sample being aligned with the axis of the antenna and without even needing to know the orientation of the material parameters of a biaxial sample to be able to accurately extract their values.

To help mitigate the errors and fluctuations experienced in the results, the following can be implemented. For angles of rotation, a digital angulometer could be used to help decrease inaccuracies with material rotations/placement. To help improve the measurements of the sample thickness, a device that can measure the thickness of the sample’s center could be used. Another improvement would be to take multiple measurements around the center and use the average. The FBS could also be placed in an antenna measurement chamber to remove outside noise and interference during the measurement phase. Lastly, a larger biaxial sample could be measured so that the FBS is able to operate within it’s designed frequencies range. With these methods, the fluctuations in the results could be significantly reduced.

7.1 Future Research

For a continuation on the work presented in this thesis, the same methods can be applied using magnetic materials to test how well the calculations work when both permittivity and permeability are unknown. Another continuation would be to expand these methods to deal with samples of greater complexity then biaxial materials to see if a similar methodology can be used to extract the material parameters.
Appendix A. Background Definitions

FBS = focus beam system
EM = electromagnetic
E = electric Field
H = magnetic field
B = magnetic flux density
D = electric flux density
f = frequency
\( \omega \) = angular frequency
\( \rho_e \) = electric charge density
\( \rho_m \) = magnetic charge density
dp\(_i\) = partial electric dipole
P = electric dipole moment
Q = charge
D\(_0\) = electric flux density of free space
B\(_0\) = magnetic flux density of free space
dm\(_i\) = Partial Magnetic Dipole
M = magnetic polarization vector
I = Current
\( \hat{n} \) = Normal Vector
Appendix B. Material Property Notation

Permittivity

\( \mathbf{\varepsilon} \) = permittivity tensor

\( \varepsilon_0 \) = permittivity of free space

\( \varepsilon_r \) = relative permittivity

\( \varepsilon_\theta \) = permittivity from rotation around Y axis

\( \varepsilon_\phi \) = permittivity from rotation around Z axis

\( \varepsilon_{xx} \) = permittivity that effects an E field entering in a X orientation and outputs a X orientated E field.

\( \varepsilon_{xy} \) = permittivity that effects an E field entering in a Y orientation and outputs a X orientated E field.

\( \varepsilon_{xz} \) = permittivity that effects an E field entering in a Z orientation and outputs a X orientated E field.

\( \varepsilon_{yx} \) = permittivity that effects an E field entering in a X orientation and outputs a Y orientated E field.

\( \varepsilon_{yy} \) = permittivity that effects an E field entering in a Y orientation and outputs a Y orientated E field.

\( \varepsilon_{yz} \) = permittivity that effects an E field entering in a Z orientation and outputs a Y orientated E field.

\( \varepsilon_{zx} \) = permittivity that effects an E field entering in a X orientation and outputs a Z orientated E field.

\( \varepsilon_{zy} \) = permittivity that effects an E field entering in a Y orientation and outputs a Z orientated E field.

\( \varepsilon_{zz} \) = permittivity that effects an E field entering in a Z orientation and outputs a Z orientated E field.

\( \varepsilon^{\theta}_{x'x'} \) = permittivity that effects an E field entering in a X orientation and outputs a X orientated E field for a sample rotated around the Y axis

\( \varepsilon^{\theta}_{x'y'} \) = permittivity that effects an E field entering in a Y orientation and outputs a X orientated E field for a sample rotated around the Y axis

\( \varepsilon^{\theta}_{x'z'} \) = permittivity that effects an E field entering in a Z orientation and outputs a X orientated E field for a sample rotated around the Y axis
\(\epsilon^\theta_{y'x'}\) = permittivity that effects an E field entering in a X orientation and outputs a Y orientated E field for a sample rotated around the Y axis

\(\epsilon^\theta_{y'y'}\) = permittivity that effects an E field entering in a Y orientation and outputs a Y orientated E field for a sample rotated around the Y axis

\(\epsilon^\theta_{y'z'}\) = permittivity that effects an E field entering in a Z orientation and outputs a Y orientated E field for a sample rotated around the Y axis

\(\epsilon^{\phi}_{z'x'}\) = permittivity that effects an E field entering in a X orientation and outputs a Z orientated E field for a sample rotated around the Y axis

\(\epsilon^{\phi}_{z'y'}\) = permittivity that effects an E field entering in a Y orientation and outputs a Z orientated E field for a sample rotated around the Y axis

\(\epsilon^{\phi}_{z'z'}\) = permittivity that effects an E field entering in a Z orientation and outputs a Z orientated E field for a sample rotated around the Y axis

\(\epsilon^\phi_{x'x'}\) = permittivity that effects an E field entering in a X orientation and outputs a X orientated E field for a sample rotated around the Z axis

\(\epsilon^\phi_{x'y'}\) = permittivity that effects an E field entering in a Y orientation and outputs a X orientated E field for a sample rotated around the Z axis

\(\epsilon^\phi_{x'z'}\) = permittivity that effects an E field entering in a Z orientation and outputs a X orientated E field for a sample rotated around the Z axis

\(\epsilon^\phi_{y'x'}\) = permittivity that effects an E field entering in a X orientation and outputs a Y orientated E field for a sample rotated around the Z axis

\(\epsilon^\phi_{y'y'}\) = permittivity that effects E field entering in a Y orientation and outputs Y orientated E fields for a sample rotated around the Z axis

\(\epsilon^\phi_{y'z'}\) = permittivity that effects an E field entering in a Z orientation and outputs a Y orientated E field for a sample rotated around the Z axis

\(\epsilon^\phi_{z'x'}\) = permittivity that effects an E field entering in a X orientation and outputs a Z orientated E field for a sample rotated around the Z axis

\(\epsilon^\phi_{z'y'}\) = permittivity that effects an E field entering in a Y orientation and outputs a Z orientated E field for a sample rotated around the Z axis

\(\epsilon^\phi_{z'z'}\) = permittivity that effects an E field entering in a Z orientation and outputs a Z orientated E field for a sample rotated around the Z axis
Permeability

$\vec{\mu}$ = permeability tensor

$\mu_0$ = permeability of free space

$\vec{\mu}_r$ = relative permeability

$\mu_{xx}$ = permeability that effects an H field entering in a X orientation and outputs a X orientated H field.

$\mu_{xy}$ = permeability that effects an H field entering in a Y orientation and outputs a X orientated H field.

$\mu_{xz}$ = permeability that effects an H field entering in a Z orientation and outputs a X orientated H field.

$\mu_{yx}$ = permeability that effects an H field entering in a X orientation and outputs a Y orientated H field.

$\mu_{yy}$ = permeability that effects an H field entering in a Y orientation and outputs a Y orientated H field.

$\mu_{yz}$ = permeability that effects an H field entering in a Z orientation and outputs a Y orientated H field.

$\mu_{zx}$ = permeability that effects an H field entering in a X orientation and outputs a Z orientated H field.

$\mu_{zy}$ = permeability that effects an H field entering in a Y orientation and outputs a Z orientated H field.

$\mu_{zz}$ = permeability that effects an H field entering in a Z orientation and outputs a Z orientated H field.

$\mu_{x'x''}^\theta$ = permeability that effects an H field entering in a X orientation and outputs a X orientated H field for a sample rotated around the Y axis

$\mu_{y'y''}^\theta$ = permeability that effects an H field entering in a Y orientation and outputs a X orientated H field for a sample rotated around the Y axis

$\mu_{z'z''}^\theta$ = permeability that effects an H field entering in a Z orientation and outputs a X orientated H field for a sample rotated around the Y axis
\( \mu_{y'z'}^\theta \) = permeability that effects an H field entering in a X orientation and outputs a Y orientated H field for a sample rotated around the Y axis

\( \mu_{y'y'}^\theta \) = permeability that effects an H field entering in a Y orientation and outputs a Y orientated H field for a sample rotated around the Y axis

\( \mu_{y'z'}^\theta \) = permeability that effects an H field entering in a Z orientation and outputs a Y orientated H field for a sample rotated around the Y axis

\( \mu_{z'z'}^\theta \) = permeability that effects an H field entering in a X orientation and outputs a Z orientated H field for a sample rotated around the Y axis

\( \mu_{z'y'}^\theta \) = permeability that effects an H field entering in a Y orientation and outputs a Z orientated H field for a sample rotated around the Y axis

\( \mu_{z'z'}^\phi \) = permeability that effects an H field entering in a Z orientation and outputs a Z orientated H field for a sample rotated around the Y axis

\( \mu_{x'x'}^\phi \) = permeability that effects an H field entering in a X orientation and outputs a X orientated H field for a sample rotated around the Z axis

\( \mu_{x'y'}^\phi \) = permeability that effects an H field entering in a Y orientation and outputs a X orientated H field for a sample rotated around the Z axis

\( \mu_{x'z'}^\phi \) = permeability that effects an H field entering in a Z orientation and outputs a X orientated H field for a sample rotated around the Z axis

\( \mu_{y'x'}^\phi \) = permeability that effects an H field entering in a Y orientation and outputs a Y orientated H field for a sample rotated around the Z axis

\( \mu_{y'y'}^\phi \) = permeability that effects an H field entering in a Y orientation and outputs Y orientated H fields for a sample rotated around the Z axis

\( \mu_{y'z'}^\phi \) = permeability that effects an H field entering in a Z orientation and outputs Y orientated H field for a sample rotated around the Z axis

\( \mu_{x'y'}^\phi \) = permeability that effects an H field entering in a Y orientation and outputs Y orientated H field for a sample rotated around the Z axis

\( \mu_{x'z'}^\phi \) = permeability that effects an H field entering in a Z orientation and outputs Y orientated H field for a sample rotated around the Z axis

\( \mu_{z'x'}^\phi \) = permeability that effects an H field entering in a X orientation and outputs Z orientated H field for a sample rotated around the Z axis

\( \mu_{z'y'}^\phi \) = permeability that effects an H field entering in a Y orientation and outputs Z orientated H field for a sample rotated around the Z axis

\( \mu_{z'z'}^\phi \) = permeability that effects an H field entering in a Z orientation and outputs Z orientated H field for a sample rotated around the Z axis
Appendix B. Thesis Definitions

$\hat{K}$ = propagation constant in direction of travel
$K_x$ = propagation constant along the X axis
$K_z$ = propagation constant along the Z axis
$\hat{T}_y$ = transformation matrix for rotation around the Y axis
$\hat{T}_z$ = transformation matrix for rotation around the Z axis
$\hat{T}_{zy}$ = transformation matrix for rotation around the Y and Z axes
Appendix C: S Parameters

Non-Directional

\( S_{11} \) = the portion of the EM wave traveling from Antenna 1, reflecting off of the sample, and returning to antenna 1.

\( S_{22} \) = the portion of the EM wave traveling from Antenna 2, reflecting off of the sample, and returning to antenna 2.

\( S_{21} \) = the portion of the EM wave from antenna 1, transmitted through the sample and received by antenna 2.

\( S_{12} \) = the portion of the EM wave from antenna 2 transmitted through the sample and received by antenna 1.

Directional Dependence

\( S_{xx}^{11} \) = the portion of the EM wave from antenna 1 in the X directed orientation, reflected off of the sample, and returned to antenna 1 in the X directed orientation.

\( S_{yy}^{11} \) = the portion of the EM wave from antenna 1 in the Y directed orientation, reflected off of the sample, and returned to antenna 1 in the Y directed orientation.

\( S_{xx}^{21} \) = the portion of the EM wave from antenna 1 in the X directed orientation, transmitted through the sample, and received by antenna 2 in the X directed orientation.

\( S_{yy}^{21} \) = the portion of the EM wave from antenna 1 in the Y directed orientation, transmitted through the sample, and returned to antenna 1 in the Y directed orientation.

\( S_{xx}^{12} \) = the portion of the EM wave from antenna 2 in the X directed orientation, transmitted through the sample, and received by antenna 1 in the X directed orientation.

\( S_{yy}^{12} \) = the portion of the EM wave from antenna 2 in the Y directed orientation, transmitted through the sample, and received by antenna 1 in the Y directed orientation.

\( S_{xx}^{22} \) = the portion of the EM wave from antenna 2 in the X directed orientation, reflected off of the sample, and returned to antenna 2 in the X directed orientation.

\( S_{yy}^{22} \) = the portion of the EM wave from antenna 2 in the Y directed orientation, reflected off of the sample, and returned to antenna 2 in the Y directed orientation.
**Cross-Polarized**

\[ S_{11}^{xy} = \text{the portion of the EM wave from antenna 1 in the Y directed orientation, reflected off of the sample, and returned to antenna 1 in the X directed orientation.} \]

\[ S_{11}^{yx} = \text{the portion of the EM wave from antenna 1 in the X directed orientation, reflected off of the sample, and returned to antenna 1 in the Y directed orientation.} \]

\[ S_{21}^{xy} = \text{the portion of the EM wave from antenna 1 in the Y directed orientation, transmitted through the sample, and received by antenna 2 in the X directed orientation.} \]

\[ S_{21}^{yx} = \text{the portion of the EM wave from antenna 1 in the X directed orientation, transmitted through the sample, and returned to antenna 1 in the Y directed orientation.} \]

\[ S_{12}^{xy} = \text{the portion of the EM wave from antenna 2 in the Y directed orientation, transmitted through the sample, and received by antenna 1 in the X directed orientation.} \]

\[ S_{12}^{yx} = \text{the portion of the EM wave from antenna 2 in the X directed orientation, transmitted through the sample, and received by antenna 1 in the Y directed orientation.} \]

\[ S_{22}^{xy} = \text{the portion of the EM wave from antenna 2 in the Y directed orientation, reflected off of the sample, and returned to antenna 2 in the X directed orientation.} \]

\[ S_{22}^{yx} = \text{the portion of the EM wave from antenna 2 in the X directed orientation, reflected off of the sample, and returned to antenna 2 in the Y directed orientation.} \]

**Sections**

\[ S_{21}^A = \text{The S parameter correlating to the space between antenna 1 and the beginning of the sample traveling from antenna 1 to the sample.} \]

\[ S_{12}^A = \text{The S parameter correlating to the space between antenna 1 and the beginning of the sample traveling from the sample to antenna 1.} \]

\[ S_{21}^B = \text{The S parameter correlating to the space between antenna 2 and the beginning of the sample traveling from the sample to antenna 2.} \]

\[ S_{12}^B = \text{The S parameter correlating to the space between antenna 2 and the beginning of the sample traveling from antenna 2 to the sample.} \]

\[ S_{21}^S = \text{The S parameter correlating to the sample and traveling in the direction of antenna 1 to antenna 2.} \]
Directional Sections

\( S_{21}^{xxA} \) = The S parameter correlating to the space between antenna 1 and the beginning of the sample traveling from antenna 1 to the sample with the E field in the X direction.

\( S_{21}^{yyA} \) = The S parameter correlating to the space between antenna 1 and the beginning of the sample traveling from antenna 1 to the sample with the E field in the Y direction.

\( S_{12}^{xxA} \) = The S parameter correlating to the space between antenna 1 and the beginning of the sample traveling from the sample to antenna 1 with the E field in the X direction.

\( S_{12}^{yyA} \) = The S parameter correlating to the space between antenna 1 and the beginning of the sample traveling from the sample to antenna 1 with the E field in the Y direction.

\( S_{21}^{xxB} \) = The S parameter correlating to the space between the end of the sample and antenna 2 traveling from the sample to antenna 2 with the E field in the X direction.

\( S_{21}^{yyB} \) = The S parameter correlating to the space between the end of the sample and antenna 2 traveling from the sample to antenna 2 with the E field in the Y direction.

\( S_{12}^{xxB} \) = The S parameter correlating to the space between the end of the sample and antenna 2 traveling from antenna 2 to the sample with the E field in the X direction.

\( S_{12}^{yyB} \) = The S parameter correlating to the space between the end of the sample and antenna 2 traveling from antenna 2 to the sample with the E field in the Y direction.

\( S_{21}^{xxS} \) = The S parameter correlating to the sample and traveling in the direction of antenna 1 to antenna 2 with the E field in the X direction.

\( S_{21}^{yyS} \) = The S parameter correlating to the sample and traveling in the direction of antenna 1 to antenna 2 with the E field in the Y direction.

\( S_{12}^{xxS} \) = The S parameter correlating to the sample and traveling in the direction of antenna 2 to antenna 1 with the E field in the X direction.

\( S_{12}^{yyS} \) = The S parameter correlating to the sample and traveling in the direction of antenna 2 to antenna 1 with the E field in the Y direction.
Non-Directional Measurements

$S_{21}^{MT}$ = represents the entire space between antenna 1 and antenna 2 with a sample and the EM wave traveling from antenna 1 to antenna 2

$S_{12}^{MS}$ = represents the entire space between antenna 1 and antenna 2 with a sample and the EM wave traveling from antenna 2 to antenna 1

$S_{11}^{MS}$ = represents the entire space of the EM wave traveling from antenna 1 reflecting off the sample, and traveling back to antenna 1.

$S_{22}^{MS}$ = represents the entire space of the EM wave traveling from antenna 2 reflecting off the sample, and traveling back to antenna 2.

$S_{21}^{MT}$ = represents the entire space between antenna 1 and antenna 2 without a sample and the EM wave traveling from antenna 1 to antenna 2

$S_{12}^{MT}$ = represents the entire space between antenna 1 and antenna 2 without a sample and the EM wave traveling from antenna 2 to antenna 1
Directional Measurements

$S_{21}^{xx MS} = \text{represents the entire space between antenna 1 and antenna 2 with a sample and the EM wave emitted from antenna 1 in the X direction and received by antenna 2 in the X direction}$

$S_{21}^{yy MS} = \text{represents the entire space between antenna 1 and antenna 2 with a sample and the EM wave emitted from antenna 1 in the Y direction and received by antenna 2 in the Y direction}$

$S_{12}^{xx MS} = \text{represents the entire space between antenna 1 and antenna 2 with a sample and the EM wave emitted from antenna 2 in the X direction and received by antenna 1 in the X direction}$

$S_{12}^{yy MS} = \text{represents the entire space between antenna 1 and antenna 2 with a sample and the EM wave emitted from antenna 2 in the Y direction and received by antenna 1 in the Y direction}$

$S_{11}^{xx MS} = \text{represents the entire space of the E field emitted from antenna 1 in the X direction reflecting off the sample, and received by antenna 1 in the X direction}$

$S_{11}^{yy MS} = \text{represents the entire space of the E field emitted from antenna 1 in the y direction reflecting off the sample, and received by antenna 1 in the y direction}$

$S_{22}^{xx MS} = \text{represents the entire space of the E field emitted from antenna 2 in the X direction reflecting off the sample, and received by antenna 2 in the X direction}$

$S_{22}^{yy MS} = \text{represents the entire space of the E field emitted from antenna 2 in the X direction reflecting off the sample, and received by antenna 2 in the X direction}$

$S_{21}^{xx MT} = \text{represents the entire space between antenna 1 and antenna 2 without a sample and the E field traveling from antenna 1 to antenna 2 in the X direction}$

$S_{21}^{yy MT} = \text{represents the entire space between antenna 1 and antenna 2 without a sample and the E field traveling from antenna 1 to antenna 2 in the Y direction}$

$S_{12}^{xx MT} = \text{represents the entire space between antenna 1 and antenna 2 without a sample and the E field traveling from antenna 2 to antenna 1 in the X direction}$

$S_{12}^{yy MT} = \text{represents the entire space between antenna 1 and antenna 2 without a sample and the E field traveling from antenna 2 to antenna 1 in the Y direction}$
Y Axis Rotated Sample

\( S_{\theta}^{x} \), \( S_{\theta}^{x} \)

= S parameter tensor or a sample rotated around Y axis

\( S_{\theta}^{x} \), \( S_{\theta}^{x} \)

= S parameter for X directed E field entering and leaving along the X axis for a sample rotated around the Y axis

\( S_{\theta}^{z} \), \( S_{\theta}^{z} \)

= S parameter for E field entering along the Z axis and exiting along the X axis for a sample rotated around the Y axis.

\( S_{\theta}^{x} \), \( S_{\theta}^{x} \)

= S parameter for Z directed E field entering and leaving along the Z axis for a sample rotated around the Y axis.

\( S_{\theta}^{x} \), \( S_{\theta}^{x} \)

= S parameter for E field entering along the X axis and exiting along the Z axis for a sample rotated around the Y axis.
Z Axis Rotated Sample

\[ S_{\phi}^{S} \] = S parameter tensor or a sample rotated around Z axis \( \phi \) degrees

\[ S_{\phi}^{x'x'} \] = S parameter for X directed E field entering and leaving along the X axis for a sample rotated around the Z axis \( \phi \) degrees

\[ S_{\phi}^{y'y'} \] = S parameter for X directed E field entering and leaving along the X axis for a sample rotated around the Y axis \( \phi \) degrees

\[ S_{\phi}^{z'x'} \] = S parameter for E field entering along the Z axis and exiting along the X axis for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{\phi}^{y'x'} \] = S parameter for E field entering along the X axis and exiting along the Y axis for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{\phi}^{y'y'} \] = S parameter for E field entering along the Y axis and exiting along the Y axis for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{\phi}^{y'z'} \] = S parameter for E field entering along the Z axis and exiting along the Y axis for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{\phi}^{z'x'} \] = S parameter for Z directed E field entering and leaving along the Z axis for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{\phi}^{z'y'} \] = S parameter for X directed E field traveling from antenna 1, reflecting off the sample, and the E field returning to antenna 1 in the X direction for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{\phi}^{z'z'} \] = S parameter for X directed E field traveling from antenna 1 transmitting through the sample and traveling to antenna 2 in the X direction for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{\phi}^{21} \] = S parameter for X directed E field traveling from antenna 2, reflecting off the sample, and the E field returning to antenna 2 in the X direction for a sample rotated around the Z axis \( \phi \) degrees.
$S_{11}^{x'y'} = S$ parameter for Y directed E field traveling from antenna 1, reflecting off the sample, and the E field returning to antenna 1 in the X direction for a sample rotated around the Z axis $\phi$ degrees.

$S_{12}^{x'y'} = S$ parameter for Y directed E field traveling from antenna 1 transmitting through the sample and traveling to antenna 2 in the X direction for a sample rotated around the Z axis $\phi$ degrees.

$S_{21}^{x'y'} = S$ parameter for Y directed E field traveling from antenna 1 transmitting through the sample and traveling to antenna 2 in the X direction for a sample rotated around the Z axis $\phi$ degrees.

$S_{22}^{x'y'} = S$ parameter for Y directed E field traveling from antenna 2, reflecting off the sample, and the E field returning to antenna 2 in the X direction for a sample rotated around the Z axis $\phi$ degrees.

$S_{11}^{y'x'} = S$ parameter for X directed E field traveling from antenna 1, reflecting off the sample, and the E field returning to antenna 1 in the Y direction for a sample rotated around the Z axis $\phi$ degrees.

$S_{12}^{y'x'} = S$ parameter for X directed E field traveling from antenna 1 transmitting through the sample and traveling to antenna 2 in the Y direction for a sample rotated around the Z axis $\phi$ degrees.

$S_{21}^{y'x'} = S$ parameter for X directed E field traveling from antenna 2, reflecting off the sample, and the E field returning to antenna 2 in the Y direction for a sample rotated around the Z axis $\phi$ degrees.

$S_{22}^{y'x'} = S$ parameter for X directed E field traveling from antenna 2 transmitting through the sample and traveling to antenna 2 in the Y direction for a sample rotated around the Z axis $\phi$ degrees.

$S_{11}^{z'x'} = S$ parameter for Z directed E field traveling from antenna 1, reflecting off the sample, and the E field returning to antenna 1 in the X direction for a sample rotated around the Z axis $\phi$ degrees.

$S_{12}^{z'x'} = S$ parameter for Z directed E field traveling from antenna 1 transmitting through the sample and traveling to antenna 2 in the X direction for a sample rotated around the Z axis $\phi$ degrees.
\[ S_{21,\phi}^{x'}z' \] = S parameter for Z directed E field traveling from antenna 1 transmitting through the sample and traveling to antenna 2 in the X direction for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{22,\phi}^{x'}z' \] = S parameter for Z directed E field traveling from antenna 2, reflecting off the sample, and the E field returning to antenna 2 in the X direction for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{11,\phi}^{z'}z' \] = S parameter for Z directed E field traveling from antenna 1, reflecting off the sample, and the E field returning to antenna 1 in the Z direction for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{12,\phi}^{z'}z' \] = S parameter for Z directed E field traveling from antenna 1 transmitting through the sample and traveling to antenna 2 in the Z direction for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{21,\phi}^{z'}z' \] = S parameter for Z directed E field traveling from antenna 1 transmitting through the sample and traveling to antenna 2 in the Z direction for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{22,\phi}^{z'}z' \] = S parameter for X directed E field traveling from antenna 2, reflecting off the sample, and the E field returning to antenna 2 in the Z direction for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{11,\phi}^{z'}x' \] = S parameter for X directed E field traveling from antenna 1, reflecting off the sample, and the E field returning to antenna 1 in the Z direction for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{12,\phi}^{z'}x' \] = S parameter for X directed E field traveling from antenna 1 transmitting through the sample and traveling to antenna 2 in the Z direction for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{21,\phi}^{z'}x' \] = S parameter for X directed E field traveling from antenna 1 transmitting through the sample and traveling to antenna 2 in the Z direction for a sample rotated around the Z axis \( \phi \) degrees.

\[ S_{22,\phi}^{z'}x' \] = S parameter for X directed E field traveling from antenna 2, reflecting off the sample, and the E field returning to antenna 2 in the Z direction for a sample rotated around the Z axis \( \phi \) degrees.
Second Sample

\( S_{21}^{xxS2} \) = The S parameter correlating to the sample with known axis orientation and traveling in the direction of antenna 1 to antenna 2 with the E field in the X direction.

\( S_{21}^{yyS2} \) = The S parameter correlating to the sample with known axis orientation and traveling in the direction of antenna 1 to antenna 2 with the E field in the Y direction.

\( S_{21}^{xxMS2} \) = represents the entire space between antenna 1 and antenna 2 with a sample with known axis orientation and the EM wave emitted from antenna 1 in the X direction and received by antenna 2 in the X direction.

\( S_{21}^{yyMS2} \) = represents the entire space between antenna 1 and antenna 2 with the sample with known axis orientation and the EM wave emitted from antenna 1 in the Y direction and received by antenna 2 in the Y direction.

\( S_{21}^{xyMS2} \) = represents the entire space between antenna 1 and antenna 2 with a sample with known axis orientation and the EM wave emitted from antenna 1 in the Y direction and received by antenna 2 in the X direction.

\( S_{21}^{yxMS2} \) = represents the entire space between antenna 1 and antenna 2 with the sample with known axis orientation and the EM wave emitted from antenna 1 in the X direction and received by antenna 2 in the Y direction.

\( S_{21}^{y'x'S2} \) = \( S \) parameter for X directed E field traveling from antenna 1 transmitting through the sample and traveling to antenna 2 in the Y direction for a sample rotated around the Z axis \( \phi_2 \) degrees.

\( S_{21}^{y'y'S2} \) = \( S \) parameter for Y directed E field traveling from antenna 1 transmitting through the sample and traveling to antenna 2 in the X direction for a sample rotated around the Z axis \( \phi_2 \) degrees.
Y and Z Axis Rotated Sample

\[ S'_{\theta\phi} = \text{S parameter tensor for sample rotated around the Z and Y axes.} \]

\[ S^{x'x'}_{\theta\phi} = \text{S parameter for X directed E field entering a sample, rotated around the Z and Y axes, and leaving as a X directed E field.} \]

\[ S^{x'y'}_{\theta\phi} = \text{S parameter for Y directed E field entering a sample, rotated around the Z and Y axes, and leaving as a X directed E field.} \]

\[ S^{x'z'}_{\theta\phi} = \text{S parameter for Z directed E field entering a sample, rotated around the Z and Y axes, and leaving as a X directed E field.} \]

\[ S^{y'x'}_{\theta\phi} = \text{S parameter for X directed E field entering a sample, rotated around the Z and Y axes, and leaving as a Y directed E field.} \]

\[ S^{y'y'}_{\theta\phi} = \text{S parameter for Y directed E field entering a sample, rotated around the Z and Y axes, and leaving as a Y directed E field.} \]

\[ S^{y'z'}_{\theta\phi} = \text{S parameter for Z directed E field entering a sample, rotated around the Z and Y axes, and leaving as a Y directed E field.} \]

\[ S^{x'x'}_{\theta\phi} = \text{S parameter for X directed E field entering a sample, rotated around the Z and Y axes, and leaving as a Z directed E field.} \]

\[ S^{x'y'}_{\theta\phi} = \text{S parameter for Y directed E field entering a sample, rotated around the Z and Y axes, and leaving as a Z directed E field.} \]

\[ S^{x'z'}_{\theta\phi} = \text{S parameter for Z directed E field entering a sample, rotated around the Z and Y axes, and leaving as a Z directed E field.} \]

\[ S^{z'y'}_{11\theta\phi} = \text{S parameter for Y directed E field emitted from antenna 1, hitting a sample that is rotated around the Z and Y axes, being reflected, and returning to antenna 1 as a X directed E field.} \]

\[ S^{z'y'}_{22\theta\phi} = \text{S parameter for Y directed E field emitted from antenna 2, hitting a sample that is rotated around the Z and Y axes, being reflected, and returning to antenna 2 as a X directed E field.} \]

\[ S^{z'y'}_{21\theta\phi} = \text{S parameter for Y directed E field emitted from antenna 1, traveling through a sample that is rotated around the Z and Y axes and being received by antenna 2 as a X directed E field.} \]
$S_{12\theta\phi}^{x'y'}$ = S parameter for Y directed E field emitted from antenna 2, traveling through a sample that is rotated around the Z and Y axes and being received by antenna 1 as a X directed E field.

$S_{11\theta\phi}^{y'x'}$ = S parameter for X directed E field emitted from antenna 1, hitting a sample that is rotated around the Z and Y axes, being reflected, and returning to antenna 1 as a Y directed E field.

$S_{22\theta\phi}^{y'x'}$ = S parameter for X directed E field emitted from antenna 2, hitting a sample that is rotated around the Z and Y axes, being reflected, and returning to antenna 2 as a Y directed E field.

$S_{21\theta\phi}^{y'x'}$ = S parameter for X directed E field emitted from antenna 1, traveling through a sample that is rotated around the Z and Y axes and being received by antenna 2 as a Y directed E field.

$S_{12\theta\phi}^{y'x'}$ = S parameter for X directed E field emitted from antenna 2, traveling through a sample that is rotated around the Z and Y axes and being received by antenna 1 as a Y directed E field.
Bibliography


Focus Beam System Biaxial Material Characterization

O'Gorman, Nicholas A, 2nd Lt.

In this thesis paper, a method for extracting the material parameters for a biaxial material using a focus beam system is derived and tested, including for the case when the orientation of the sample’s axes are unknown. The results are then compared to those obtained utilizing other methods with the same material design, verifying that the methodologies work.

Electromagnetics, Material Characterization, Metamaterials, Focus Beam System

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<td>17. LIMITATION OF ABSTRACT</td>
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<tr>
<td>19a. NAME OF RESPONSIBLE PERSON</td>
<td>Dr. Michael J. Havrilla, AFIT/ENG</td>
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<tr>
<td>19b. TELEPHONE NUMBER (include area code)</td>
<td>(312) 785-3636 x4582; <a href="mailto:Michael.Havrilla@afit.edu">Michael.Havrilla@afit.edu</a></td>
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