Subaperture sampling for digital-holography applications involving atmospheric turbulence

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Using wave-optics simulations, this paper defines what subaperture sampling effectively means for digital-holography applications involving atmospheric turbulence. Throughout, we consider the on-axis phase shifting recording geometry (PSRG) and off-axis PSRG, both with the effects of sensor noise. The results ultimately show that (1) insufficient subaperture sampling manifests as an efficiency loss that limits the achievable signal-to-noise ratio and field-estimated Strehl ratio; (2) digital-holography applications involving atmospheric turbulence require at least three focal-plane array (FPA) pixels per Fried coherence length to meet the Maréchal criterion; and (3) off-axis PSRG is a valid and efficient implementation with minor losses, as compared to on-axis PSRG. Such results will inform future research efforts on how to efficiently use the available FPA pixels.

1. INTRODUCTION

With the use of a strong reference, digital holography enables deep-turbulence wavefront sensing [1–3]. In practice, the strong reference boosts the highly scintillated signal above the noise floor of the camera. Digital holography, as a result, potentially enables a shot-noise limited detection regime [4], which directly combats the low signal-to-noise ratios (SNRs) that often arise in deep-turbulence conditions.

Given this enabling technology, it is not as straightforward to define what subaperture sampling means for digital-holography wavefront sensing as it is for Shack–Hartmann wavefront sensing. In the aforementioned case, there are lenslets with a defined aperture or “subaperture,” which effectively sample the incoming aberrated light [5]. Welsh and Gardner were the first to show that in the presence of atmospheric turbulence (in particular, isoplanatic phase errors), one Shack–Hartmann subaperture per Fried coherence length results in Strehl ratios of approximately 0.8 [6], therefore satisfying the Maréchal criterion [7].

A recent paper made use of Welsh and Gardner’s definition for subaperture sampling when performing compensated-beacon adaptive optics [8]. To directly compare the performance of Shack–Hartmann wavefront sensing to digital-holography wavefront sensing, the analysis also made use of the preliminary results by Banet and Spencer in a recent conference proceeding [9]. They showed that one needs at least three pixels per Fried coherence length to satisfy the Maréchal criterion and obtain Strehl ratios of approximately 0.8, using the extended Maréchal approximation [10–13]. In turn, Ref. [9] effectively defined what subaperture sampling means for digital-holography wavefront sensing.

However, Ref. [9] did not include the effects of sensor noise, nor did it formulate performance as a function of the efficiency losses that limit the achievable SNR and field-estimated Strehl ratio [14–17]. Thus, this paper builds on the preliminary results contained in Ref. [9] and, in so doing, broadens the applicability to all digital-holography applications involving atmospheric turbulence.

In what follows, we consider both on-axis and off-axis recording geometries. For this purpose, we make use of the on-axis phase shifting recording geometry (PSRG) [18]. We also explore a relatively new concept: off-axis PSRG. In comparison, off-axis PSRG requires only one digital hologram, whereas on-axis PSRG requires two to four digital holograms (preferably four [3]). Thus, in an effort to flesh out the details in a comprehensive way, this paper includes results for both recording geometries (i.e., on-axis PSRG and off-axis PSRG), so that future research efforts can readily extend the results contained herein.

In this paper, we effectively show (using wave-optics simulations) that even in the presence of sensor noise, for both on-axis and off-axis PSRGs, insufficient subaperture sampling manifests as an efficiency loss that limits the achievable SNR and field-estimated Strehl ratio. Furthermore, we definitively show that digital-holography applications involving atmospheric turbulence require at least three focal-plane array (FPA) pixels per Fried coherence length to meet the Maréchal criterion.
Such results effectively define what subaperture sampling means for digital-holography applications such as deep-turbulence wavefront sensing [1–3,19–21], in addition to branch-point-tolerant phase compensation [22–24], 3D imaging [25–27], phased-array imaging [28–30], long-range imaging with isoplanatic phase errors [31–36], and long-range imaging with anisoplanatic phase errors [37–42]. These applications all involve atmospheric turbulence and require a careful balance between spatial and temporal resolution. We emphasize this last point because using more FPA pixels, in practice, results in slower camera-read-out times. Thus, any increase in spatial resolution, gained using more FPA pixels, is thereafter met with a decrease in temporal resolution, which is not ideal when dealing with dynamic phase errors, such as those associated with atmospheric turbulence. The results contained herein will inform future research efforts on how to efficiently use digital-holography applications involving atmospheric turbulence. They will also inform other digital-holography applications such as imaging through fog [43], foliage [44], water [45], and tissue [46].

In Section 2, we formulate models for both on-axis and off-axis PSRGs. We also point out the main difference between these recording geometries and formulate the additional models needed to understand how subaperture sampling manifests for digital-holography applications involving atmospheric turbulence. Section 3 then provides the simulation setup and metrics needed to quantify performance. The results follow thereafter in Section 4 with a conclusion in Section 5.

2. MODEL FORMULATION

Figure 1 provides an illustration of the digital-holography setup. Here, we split the master oscillator (MO) laser into two paths: the illuminator and the local oscillator (LO). The illuminator flood illuminates the unresolved object, and we assume that the scattered light propagates through the atmosphere as an aberrated spherical wave. In turn, the circular pupil collimates this aberrated spherical wave to provide the signal, $U_S$. The LO provides the reference, $U_R$. As such, the box denoted PSRG contains the optics required to interfere an image of the signal with the appropriate reference to obtain the corresponding digital hologram, $i_H$, after digitization with the FPA pixels. These FPA pixels, in practice, are part of one or more cameras, depending on the recording geometry of interest. With this setup in mind, we model two variants of the PSRG, which we differentiate as on-axis and off-axis models in the ensuing sections.

![Illustration of the digital-holography setup.](image)

Fig. 1. Illustration of the digital-holography setup.

A. On-Axis Model

With on-axis PSRG, we use the four-step method to obtain four digital holograms. Here, we use a phase-shifted reference (from an on-axis LO) to gain access to an estimate of the signal. As illustrated in Fig. 2, we use the four-step method because it is more efficient with respect to the SNR and the field-estimated Strehl ratio than the three-step and two-step methods [3], and it compares well with off-axis PSRG. In Fig. 2, we assume that digitization with the FPA pixels occurs in parallel (i.e., at the same time on one or more cameras) to obtain four digital holograms.

Neglecting sensor noise for the time being, the on-axis digital holograms have the following form:

$$i^{(0)}_H = |U_S + U_R e^{-j/8}|^2$$
$$i^{(t/2)}_H = |U_S|^2 + |U_R|^2 + U_S U_R^* e^{j/8} + U_S^* U_R e^{-j/8},$$

(1)

where $\delta$ is the desired reference-phase shift, and $^*$ denotes complex conjugate. With increments of $\pi/2$, the four digital holograms become

$$i^{(0)}_H = |U_S|^2 + |U_R|^2 + U_S U_R + U_S^* U_R,$n
$$i^{(t/2)}_H = |U_S|^2 + |U_R|^2 + j U_S U_R^* - j U_S^* U_R,,$n
$$i^{(t)}_H = |U_S|^2 + |U_R|^2 - U_S U_R^* - U_S^* U_R,$n
$$i^{(3t/2)}_H = |U_S|^2 + |U_R|^2 - j U_S U_R^* + j U_S^* U_R.,$n

(2)

Now, we can solve for $U_S$ [47], such that

$$\hat{U}_S = 4 U_S U_R = \left( i^{(0)}_H - i^{(t)}_H \right) - j \left( i^{(t/2)}_H - i^{(3t/2)}_H \right),$$

(3)

where $\hat{U}_S$ is an estimate of $U_S$ with the assumption that $U_R$ is uniform. From $\hat{U}_S$, one can also obtain an estimate of the wrapped-phase function (cf. Fig. 2).

The accuracy of the estimate ultimately depends on the SNR. Here, we use the power definition for the SNR, $S/N$, viz.,

$$S/N = \frac{\langle |\hat{U}_S|^2 \rangle}{\langle |\hat{U}_N|^2 \rangle},$$

(4)

where $\hat{U}_N$ is the additive noise in $\hat{U}_S$, and $\langle \cdot \rangle$ denotes an ensemble average. Following the steps outlined in Refs. [3,4], Eq. (4) reduces to the following closed-form expression:

$$S/N = \frac{m_R m_S}{m_R + \sigma^2_r},$$

(5)

where $m_R$ is the mean number of reference photoelectrons, $m_S$ is the mean number of signal photoelectrons, $\sigma^2_r$ is the read-noise variance (associated with the read out integrated circuitry of the FPA), and $\eta_t$ is the total-system efficiency [3]. In practice, $\eta_t$ comprises multiplicative efficiency losses such as modulation efficiency, $\eta_m$, and coherence efficiency, $\eta_c$, as shown in the ensuing sections.

In writing Eq. (5), we assume several things. The first is that the reference is much stronger than the signal, such that $m_R \gg m_S$. We also assume that the noise from the reference is...
Fig. 2. Example of on-axis PSRG. $S/N = 10$, and (a) $i_{H}^{(0)}$, (b) $i_{H}^{(\pi/2)}$, (c) $i_{H}^{(\pi)}$, and (d) $i_{H}^{(3\pi/2)}$. We also show a comparison of the (e) truth wrapped phase with (f) estimated wrapped phase obtained from Eq. (3) using (a)–(d).

purely shot noise. In turn, we can use an additive noise model and include the effects of Gaussian-distributed read noise. Since shot noise is Poisson distributed, the variance is equal to the mean, such that the total-noise variance becomes $\bar{m}_R + \sigma_r^2$. Note that with a strong reference, we approximated the Poisson distribution with a Gaussian distribution, which enabled the additive sensor noise model. Also note that because we use the power definition for the SNR, the constant in Eq. (3) turns into 16, but the signal strength, $\bar{m}_S$, reduces by a factor of four to form the four digital holograms [cf. Eq. (2)]. We also add the total-noise variance, $\bar{m}_R + \sigma_r^2$, four times in Eq. (3). Therefore, the resultant coefficient reduces to unity in Eq. (5). Also note that we use the same SNR expression for the off-axis model, which we formulate next.

B. Off-Axis Model

With off-axis PSRG, we use a tilted reference (from an off-axis LO) and perform anamorphic compression to obtain a single digital hologram. As shown in Fig. 3, one can decompose the resulting digital hologram into phase-shifted holograms, and thereafter obtain an estimate of the signal. To do so, one must image the signal (collimated at the circular pupil) with cylindrical lenses. This imaging anamorphically compresses the signal, such that the circular pupil converts into an elliptical pupil with a width four times greater than its height (i.e., with a semi-major axis that is $4 \times$ its semi-minor axis).

The tilted reference, in practice, creates a linear phase ramp, such that the digital hologram undergoes a $2\pi$ phase change periodically (i.e., every fourth pixel in the $x$ direction). Here,
Fig. 3. Example of off-axis PSRG. (a) Anamorphically compressed signal wrapped phase, (b) tilted reference wrapped phase, and (c) resulting digital hologram, where $S/N = 10$. As highlighted in the magnified regions of (a)–(c), one can digitally extract every fourth column to obtain four phase-shifted holograms, which are analogous to the four digital holograms in Eq. (2) [cf. Figs. 2(a)–2(d)].

the reference complex-optical field, $U_R(x, y)$, takes the following form:

$$U_R(x, y) = A_R \exp \left( -j 2\pi x R \frac{x R}{\lambda f} \right),$$  

(6)

where $A_R$ is the uniform reference amplitude, $x_R$ is the $x$-coordinate shift of the reference, $\lambda$ is the reference wavelength, and $f$ is the focal length of the collimating lens. To achieve the correct tilt, the following criteria must be met:

$$\frac{x_R}{f} = \frac{\lambda}{4p},$$  

(7)

where $p$ is the width of a square FPA pixel. This linear phase ramp yields repeating columns in the digital hologram, where the average reference phase is $0$, $\pi/2$, $\pi$, and $3\pi/2$. Thereafter, one can digitally extract every fourth column to obtain four phase-shifted holograms, which are analogous to the four digital holograms in Eq. (2).

With Eq. (2) in mind, the procedure used to estimate the signal and wrapped-phase function for the off-axis PSRG is then identical to the on-axis PSRG [cf. Eq. (3)]. Additionally, the SNR expression for off-axis PSRG is identical to that obtained for on-axis PSRG [cf. Eq. (5)]. As with the on-axis PSRG, the use of this closed-form expression with off-axis PSRG assumes a reduction in signal strength by a factor of four in obtaining phase-shifted holograms. Next, we formulate the main difference between on-axis and off-axis models.

C. Modulation-Efficiency Model

With on-axis PSRG, one interferes the signal with a phase-shifted reference (cf. Fig. 2), whereas with off-axis PSRG, one interferes an anamorphically compressed signal with a tilted reference (cf. Fig. 3). Thus, the on-axis model digitizes four digital holograms, while the off-axis model digitizes a single digital hologram. In the latter case, one ends up modulating the digital hologram at a spatial frequency of $0.25 \frac{p}{\lambda}^\text{cycles per pixel}$. This modulation unfortunately manifests as an efficiency loss that we refer to as the modulation efficiency, $\eta_m$.

To quantify the effects of $\eta_m$, one can use the pixel modulation transfer function (MTF). As a reminder, the pixel MTF represents the spatial-frequency response of a FPA pixel. We can mathematically represent the recording of the digital hologram with the FPA as a 2D convolution between the continuous hologram, $i_H(x, y)$, and a square FPA pixel, represented as a 2D rectangle function. Using the convolution theorem, this convolution is equivalent to the 2D Fourier transform of the continuous hologram, $i_H(f_x, f_y)$, multiplied by the pixel MTF, which is a 2D sinc function (i.e., the Fourier transform of a 2D rectangle function) [48,49]. In turn,
where \( \Phi \) denotes a 2D convolution, \( p \) is again the width of a square FPA pixel, and \( F^{-1}\{[\Phi] \} \) denotes an inverse 2D Fourier transform. Since energy is conserved between both domains according to Parseval’s theorem, we quantify the effects of modulation in terms of a multiplicative loss with the 2D sinc function in the Fourier domain. Therefore, in accordance with the off-axis PSRG [cf. Eqs. (6) and (7)], the modulation is in only \( x \) direction with a spatial frequency of \( f_x = 0.25 \, p^{-1} \) and \( \eta_m = \text{sinc}^2(0.25) = 0.81 \). Note that in this case, the 1D sinc function is squared because we use the power definition for the SNR [cf. Eq. (4)]. Also note that for the on-axis PSRG, \( \eta_m = 100\% \), since there is no modulation. Thus, in the presence of modulation, \( \eta_m \) degrades due to the spatial sampling associated with the FPA pixels [3]. Spatial sampling with respect to the signal’s spatial coherence also manifests as an efficiency loss, which we formulate next.

### D. Coherence-Efficiency Model

Both on-axis and off-axis PSRGs provide an estimate of the signal. However, as the signal’s spatial coherence degrades, the accuracy of the estimate degrades due to the spatial sampling associated with the FPA pixels. To quantify the effects of this degradation, we leverage an approach originally proposed by Fried [50] (and used by Barchers and Rhodamer [19]), which develops a relationship between the phase variance, \( \sigma_{\phi}^2 \), and the number of FPA pixels per Fried coherence length, \( r_0/p \), where \( p \) is the width of the FPA pixels. As a reminder, the Fried coherence length \( r_0 \) represents the average diameter where the root-mean-square square phase error is 1.0 rad [51,52]. We then relate \( \sigma_{\phi}^2 \) and \( r_0/p \) to an efficiency loss that we refer to as the coherence efficiency, \( \eta_r \).

To develop the relationship between \( \sigma_{\phi}^2 \) and \( r_0/p \), we use a normalized signal with a unit-amplitude random field, \( u_S(x, y) \), which we simply refer to as the “truth.” In practice, the “estimate” then follows as

\[
\hat{u}_S(x_p, y_p) = \frac{1}{p^2} \int_{y_p-p/2}^{y_p+p/2} \int_{x_p-p/2}^{x_p+p/2} u_S(x, y) \, dx \, dy, \tag{9}
\]

where \( (x_p, y_p) \) are the coordinates of the FPA pixels. If we assume that the FPA pixels are square in shape, then Eq. (9) mathematically represents an average value that physically accounts for the spatial sampling associated with the FPA pixels.

To calculate the difference, \( \Delta u_S(x_p, y_p) \), between the estimate and the truth, we use the following expression:

\[
\Delta u_S(x_p, y_p) = \hat{u}_S(x_p, y_p) - u_S(x_p, y_p) = e^{i\Delta \phi_S(x_p, y_p)}, \tag{10}
\]

where \( \Delta \phi_S(x_p, y_p) \) is the phase difference between the unit-amplitude random fields. If \( \Delta \phi_S(x_p, y_p) \) follows a zero-mean Gaussian random process, then the expected value of Eq. (10) becomes

\[
\langle \Delta u_S \rangle = e^{-\sigma_{\phi}^2/2}, \tag{11}
\]

where \( \sigma_{\phi}^2 \) is the variance of \( \Delta \phi_S(x_p, y_p) \). As such,

\[
\sigma_{\phi}^2 = -2 \log \langle \Delta u_S \rangle. \tag{12}
\]

Now we need to solve for \( \langle \Delta u_S \rangle \) in terms of \( r_0/p \).

If we assume that the unit-amplitude random fields are statistically homogeneous and isotropic, then Eqs. (9) and (10) result in the following expression:

\[
\langle \Delta u_S \rangle = \frac{1}{p^2} \int_{-p/2}^{p/2} \int_{-p/2}^{p/2} \langle u_S(x, y) \rangle^2 \, dx \, dy. \tag{13}
\]

Here, we see that the integrand is equivalent to the coherence factor, \( \mu(x, y) \), from statistical optics [53]. If we assume that the atmospheric turbulence follows Kolmogorov statistics [54], then

\[
\mu(x, y) = \langle u_S(x_p + x, y_p + y) \rangle = \exp \left[ -3.44 \left( \frac{\sqrt{x^2 + y^2}}{r_0} \right)^{5/3} \right]. \tag{14}
\]

where \( r_0 \) is again the Fried coherence length. Therefore, if we let \( x = \xi \, p \) and \( y = \eta \, p \), and we substitute Eq. (14) into Eq. (13), then

\[
\langle \Delta u_S \rangle = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \exp \left[ -3.44 \left( \frac{r_0/p}{\xi^2 + \eta^2} \right)^{5/3} \right] \, d\xi \, d\eta, \tag{15}
\]

and after substitution into Eq. (12), we arrive at an integral relationship between \( \sigma_{\phi}^2 \) and \( r_0/p \).

To relate \( \sigma_{\phi}^2 \) and \( r_0/p \) to an efficiency loss, we make use of the coherence efficiency, \( \eta_r \). In previous work [16], some of the authors of this paper experimentally showed that \( \eta_r = e^{-\sigma_{\phi}^2} \) for temporal phase fluctuations. Here, we can view the efficiency loss associated with \( \sigma_{\phi}^2 \) and \( r_0/p \) as spatial phase fluctuations, so the same relationship holds true. Therefore,

\[
\eta_r = \langle \Delta u_S \rangle^2 = \left( \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \exp \left[ -3.44 \left( \frac{r_0/p}{\xi^2 + \eta^2} \right)^{5/3} \right] \, d\xi \, d\eta \right)^2, \tag{16}
\]

which relates \( r_0/p \) to an efficiency loss that limits the achievable SNR. The next section describes the simulation setup and metrics needed to quantify this last statement.

### 3. SIMULATION SETUP AND METRICS

This section describes the simulation setup needed to quantify performance. Additionally, we explain the numerical and analytical metrics used to quantify performance. These simulations make use of the wave-optics principles taught by Schmidt in MATLAB [54].

#### A. Setup

With Fig. 1 in mind, we simulated a circular pupil with a diameter \( D \) of 50 cm. This circular pupil contained a uniform-amplitude signal and isotropic phase errors that followed Kolmogorov statistics. To generate these phase errors, we used Monte Carlo phase screens. In particular, we used the Fourier transform method described by Schmidt with subharmonics [54]. The prescribed values for the Fried coherence length \( r_0 \) ranged from 0.125–5 cm; however, as discussed by Charnotskii [55], Monte Carlo phase screens are not perfect in matching
these prescribed values. Thus, we performed a coherence-factor analysis, as shown in Appendix A, comparing our numerical curves to theory [cf. Eq. (14)]. From this analysis, we found that the simulated values for \( r_0 \), on average, were \( 5.7 \pm 1.6\% \) greater than the prescribed values for \( r_0 \).

For on-axis PSRG, we set the initial grid size of the signal to \( 1,400 \times 1,400 \) grid points, and we normalized the amplitude to have a mean strength of \( \sqrt{m_S} \). The phase-shifted reference had a uniform amplitude with a mean strength of \( \sqrt{m_R} \) and a uniform phase of \( 0, \pi/2, \pi, \) and \( 3\pi/2 \) (cf. Fig. 2). Then, we reduced the signal amplitude by half and interfered with the phase-shifted reference to obtain the four digital holograms. To simulate the spatial sampling associated with the FPA pixels, we downsampled the four digital holograms to a grid size of \( 200 \times 200 \) grid points by averaging 7 \( \times \) 7 pixel bins.

For off-axis PSRG, we set the initial grid size of the signal to \( 5,600 \times 5,600 \) grid points, and we downsampled to \( 1,400 \times 5,600 \) by averaging \( 1 \times 4 \) pixel bins to simulate anamorphic compression. We normalized the signal amplitude to \( \sqrt{m_S/4} \) thereafter. Next, we set the grid size of the tilted reference to \( 1,400 \times 5,600 \) grid points with a uniform amplitude normalized to \( \sqrt{m_R} \). The tilted reference created a linear phase ramp with values of \( 0, \pi/2, \pi, \) and \( 3\pi/2 \) in every fourth column (cf. Fig. 3). Then, we interfered the signal with the reference to form a single digital hologram. To simulate the spatial sampling associated with the FPA pixels, we downsampled the digital hologram to a grid size of \( 200 \times 800 \) grid points by averaging \( 7 \times 7 \) pixel bins. In turn, we digitally extracted every fourth column from the downsampled digital hologram to create four digital holograms with a grid size of \( 200 \times 200 \) grid points.

For both on-axis and off-axis PSRGs, we simulated two sources of noise using an additive noise model. One noise source corresponded to the hologram shot noise, which followed Poisson statistics. Here, the mean \( \bar{m}_H = m_R + m_S \) was equal to the variance. Once again, note that with a strong reference, we approximated the Poisson distribution with a Gaussian distribution, which enabled the additive sensor noise model. The other noise source corresponded to the FPA read noise, which followed Gaussian statistics. Here, the variance \( \sigma_r^2 \) was 10,000 photoelectrons (pe). We also confirmed that there was no saturation of the FPA pixels, which had a well depth \( \ell \) of 100 kpe. Then, we used Eq. (3) for both on-axis and off-axis PSRGs, and we obtained an estimate of the signal, \( \hat{U}_S \).

With \( \hat{U}_S \) in mind, Table 1 shows a list of simulation parameters. For the simulation trade space, we explored a range of signal strengths, such that \( \bar{m}_S = 1–114 \text{ pe} \), which yielded ideal SNRs of \( S/N = 1–100 \). Here, the term “ideal” refers to the case where \( \eta = 100\% \) [cf. Eq. (5)]. We also explored a range of Fried coherence lengths, such that \( r_0 = 1.25 \text{ mm}–5 \text{ cm} \), which yielded spatial samplings of \( r_0/p = 0.5–20 \) with respect to the width \( p \) of the FPA pixels. In turn, we always had greater than 3.5 grid points per \( r_0 \). For each value of \( \bar{m}_S \) and \( r_0 \), we produced 50 independent realizations of noise and phase errors, concurrently. To quantify performance, we needed numerical metrics, which we formulate next.

<table>
<thead>
<tr>
<th>Table 1. Simulation Parameters</th>
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<tr>
<td><strong>Parameter</strong></td>
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<tr>
<td>Pupil diameter</td>
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<tr>
<td>Pixel width</td>
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<td>Pixel well depth</td>
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<td>Reference strength</td>
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<td>Read noise variance</td>
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<tr>
<td>Signal strength</td>
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<td>Fried’s coherence diameter</td>
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### B. Numerical Metrics

For each realization in the simulation, we calculated the numerical SNR and numerical field-estimated Strehl ratio. To calculate the numerical SNR, we needed an estimate of the noise. To obtain this noise estimate for both on-axis and off-axis PSRGs, we calculated the appropriate sums of the signal-only and reference-only irradiances (i.e., \( |\hat{U}_S(x, y)|^2 + |U_S(x, y)|^2 \)) with the additive sensor noise. Then, we propagated the appropriate sums through the simulation procedure and used Eq. (3) to produce the estimate of the noise, \( \hat{U}_N \).

With \( \hat{U}_N \) in mind, we calculated the SNR, \( S/N' \), using the following formula:

\[
S/N' = \frac{\langle |\hat{U}_S(x, y)|^2 - |\hat{U}_N(x, y)|^2 \rangle}{\langle |\hat{U}_N(x, y)|^2 \rangle}. \tag{17}
\]

Similarly, we calculated the field-estimated Strehl ratio, \( S_F' \), using the following formula:

\[
S_F' = \frac{\langle |\hat{U}_S(x, y)\hat{U}_S^*(x, y)|^2 \rangle}{\langle |\hat{U}_S(x, y)|^2 \rangle \langle |\hat{U}_S^*(x, y)|^2 \rangle}. \tag{18}
\]

One can derive Eq. (18) from the Cauchy–Schwartz inequality [3]. Therefore, values for \( S_F' \) range from zero, where the estimate of the signal is orthogonal to the truth, to one, where \( \hat{U}_S = U_S \). To compare our numerical results to analytical results, we also needed analytical metrics, which we formulate next.

### C. Analytical Metrics

For the analytical SNR, \( S/N \), we accounted for two multiplicative efficiency losses as part of the total-system efficiency, \( \eta \). These efficiency losses included: (1) modulation efficiency, \( \eta_m \), and (2) coherence efficiency, \( \eta_c \). Therefore,

\[
S/N = n_m \eta_c \frac{m_R m_S}{m_R + \sigma_r^2}. \tag{19}
\]

where for the on-axis PSRG, \( \eta_m = 100\% \); for the off-axis PSRG, \( \eta_m = 81\% \); and we numerically integrated Eq. (16) to obtain \( \eta_c \).

For the analytical field-estimated Strehl ratio, \( S_F \), we decomposed the total expression into two multiplicative terms, such that

\[
S_F = S_F^{(m)} S_F^{(c)}. \tag{20}
\]
The first term, $S_F^{(m)}$, depends on $\eta_m$ using the following relationship:

$$S_F^{(m)} = \frac{1}{1 + \frac{S}{N_m}}, \quad (21)$$

where

$$\frac{S}{N_m} = \frac{\bar{m}_R \bar{m}_S}{\bar{m}_R + \sigma^2_r}. \quad (22)$$

Note that this relationship was derived in Ref. [3], assuming that $\eta = 100\%$. Also note that Eqs. (21) and (22) have been used in previous works and have been shown to agree well with simulation [1–3]. The second term, $S_F^{(c)}$, is equivalent to $\eta_c$, viz.,

$$S_F^{(c)} = \exp \left( -\sigma^2_c \right) = \langle \Delta u_S \rangle^2 = \eta_c. \quad (23)$$

Here, we have made use of the extended Maréchal approximation [10–13]. This approximation assumes that we have statistically independent terms in Eq. (20), so that in general, variances add and Strehls multiply.

In the next section, we will compare $S/N$ [cf. Eq. (19)] to $S'/N'$ [cf. Eq. (17)]. Similarly, we will compare $S_F$ [cf. Eq. (20)] to $S_F'$ [cf. Eq. (18)]. These comparisons serve as the results for this paper.

4. RESULTS

In this section, we make use of the Maréchal criterion, which says that an optical system is approximately diffraction limited with a Strehl ratio $>0.80$ [7]. Thus, for digital-holography applications involving atmospheric turbulence, we simply say that when $S_F' > 0.8$, we have a good estimate of the signal. Recall that $S_F'$ is the numerical field-estimated Strehl ratio [cf. Eq. (18)]. Also recall that $S/N'$ is the numerical SNR [cf. Eq. (17)]. In what follows, we calculate both $S/N'$ and $S_F'$ and compare them to their analytical counterparts $S/N$ and $S_F = S_F^{(m)} S_F^{(c)}$ [cf. Eqs. (19)–(23)], respectively.

The trade space parameters for these aforementioned comparisons are $r_0/p$ and $\bar{m}_S$ (cf. Table 1). Here, $r_0/p$ provides a gauge for the spatial sampling (with respect to the signal’s spatial coherence), and $\bar{m}_S$ provides a gauge for the signal strength (which is directly proportional to $S/N_m$ when $\eta = 100\%$). First, we vary the spatial sampling, such that $r_0/p = 0.5–20$, and we show results for a strong signal, where $\bar{m}_S = 114\,\text{pe}$.

Thereafter, we vary the spatial sampling and the signal strength, such that $r_0/p = 0.5–20$ and $\bar{m}_S = 0.5–114\,\text{pe}$, and we show results for the total trade space. The results ultimately show that (1) insufficient subaperture sampling manifests as an efficiency loss that limits $S/N'$ and $S_F'$; (2) digital-holography applications involving atmospheric turbulence require $r_0/p \geq 3$ to meet the Maréchal criterion; and (3) off-axis PSRG is a valid and efficient implementation with minor losses, as compared to on-axis PSRG.

A. Strong-Signal Results

Here, we present results for one signal strength, $\bar{m}_S = 114\,\text{pe}$. In the absence of any efficiency losses due to the signal’s spatial coherence (i.e., $\eta = 100\%$), this signal strength corresponds to $S/N_m = 100$ and $S_F^{(m)} = 0.99$ for on-axis PSRG, and $S/N_m = 81$ and $S_F^{(m)} = 0.99$ for off-axis PSRG.

As shown in Fig. 4, $S/N'$ increases as $r_0/p$ increases and trends with $S/N$. The SNR difference between on-axis and off-axis models is because $\eta_m = 100\%$ for on-axis PSRG and $\eta_m = 81\%$ for off-axis PSRG. Here, $\eta_m$ is again the modulation efficiency. This outcome shows that $\eta_m$ quantifies performance as a function of $r_0/p$. For both recording geometries, $\eta_m \approx 100\%$ when $r_0/p = 20$, and $\eta_m \approx 15\%$ when $r_0/p = 0.5$.

Next we turn to Fig. 5, where both on-axis and off-axis PSRGs yield similar results for $S_F'$. Here, $\eta_m = 100\%$ for on-axis PSRG, and $\eta_m = 81\%$ for off-axis PSRG; however, we see that $\eta_m$ has only a minor effect on $S_F'$. From this observation, we believe that the off-axis PSRG is a valid and efficient implementation for digital-holography applications with only minor efficiency losses.

In general, both $S/N$ and $S_F$ slightly underestimate $S/N'$ and $S_F'$, respectively. We believe this outcome is due to two things: (1) the limitations of the extended Maréchal approximation, since this approximation is less effective with larger
B. Total-Trade-Space Results

Here, we present results for the total trade space, \( r_0/p = 0.5 - 20 \) and \( \bar{m}_S = 1-114 \, \text{pe} \). Figures 6(a) and 6(b) show results for \( S_F^e \) for on-axis and off-axis PSRGs, respectively. Note that the dotted lines denote the Maréchal criterion predicted by theory. With the Maréchal criterion in mind, if \( \bar{m}_S = 4 \, \text{pe} \) for on-axis PSRG or \( \bar{m}_S = 5 \, \text{pe} \) for off-axis PSRG, then \( S/N_m = 4 \) and \( S_F^e = 0.8 \) when \( \eta = 100\% \), hence the horizontal dotted lines. However, if \( r_0/p = 3 \), then \( S_F^e = 0.8 \), hence the vertical dotted lines. Where the dotted lines cross, \( S_F \approx S_F^e \approx 0.64 \), which signifies a poor estimate of the signal. Thus, to achieve a good estimate of the signal, we conclude that we need one of two things: (1) a stronger signal (i.e., \( S/N_m > 4 \)) when \( r_0/p = 3 \) or (2) more spatial sampling (i.e., \( r_0/p > 3 \)) when \( S/N_m = 4 \).

Figures 6(c) and 6(d) show results for the difference \( \Delta S_F \) between \( S_F^e \) and \( S_F \) for on-axis and off-axis PSRGs, respectively. In general, the simulations agree well with theory (i.e., \( \Delta S_F \approx 0 \)), except when \( r_0/p < 3 \) and \( S/N_m > 4 \). Here, \( S_F \) under estimates \( S_F^e \). These results, nonetheless, show that \( S_F \approx S_F^e \) provides a good model for estimating \( S_F^e \).

One last point worth mentioning from Fig. 6 is that the off-axis PSRG has comparable performance to the on-axis PSRG. Thus, we conclude that like the on-axis PSRG, off-axis PSRG is a viable recording geometry. Both recording geometries, in practice, enable digital-holography applications involving atmospheric turbulence.

5. CONCLUSION

The analysis contained in this paper effectively defines what subaperture sampling means for digital-holography applications involving atmospheric turbulence. Throughout, we considered two recording geometries: on-axis PSRG and off-axis PSRG, both with the effects of sensor noise. The results ultimately showed that (1) insufficient subaperture sampling manifests as an efficiency loss that limits the achievable SNR and field-estimated Strehl ratio; (2) digital-holography applications involving atmospheric turbulence require at least three FPA pixels per Fried coherence length to meet the Maréchal criterion; and (3) off-axis PSRG is a valid and efficient implementation with minor losses, as compared to on-axis PSRG. Such results will inform future research efforts on how to efficiently use the available FPA pixels.

APPENDIX A

In this appendix, we perform a coherence-factor analysis using Monte Carlo phase screens. Note that these Monte Carlo phase screens generate phase errors that follow Kolmogorov statistics. Also note that we use the Fourier transform method described by Schmidt with subharmonics [54]. As such, in Figs. 7(a) and 7(b), we show a single realization for the Monte Carlo phase screens with two different Fried coherence lengths, \( r_0 \). For each realization, we estimate the coherence factor, \( \mu(r) \), where \( r = \sqrt{x^2 + y^2} \), by numerically calculating the autocorrelation,
and thereafter the mean for 50 independent realizations. Then, we normalize the mean autocorrelation to have a maximum amplitude of one and numerically calculate an azimuthal average, as shown in Figs. 7(c) and 7(d). To compare the numerical curves to theory, we fit Eq. (14) to the numerical curves to provide a simulated value for \( r_0 \). On average, the simulated values for \( r_0 \) are \( 5.7 \pm 1.6\% \) greater than the prescribed values for \( r_0 \).

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