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Legendre G-array pairs and the theoretical unification of several G-array families

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Abstract

We investigate how Legendre G-array pairs are related to several different perfect binary G-array families. In particular we study the relations between Legendre G-array pairs, Sidelnikov-Lempel-Cohn-Eastman $\mathbb{Z}_{q-1}$-arrays, Yamada-Pott G-array pairs, Ding-Helleseth-Martinsen $\mathbb{Z}_2 \times \mathbb{Z}_p^m$-arrays, Yamada $\mathbb{Z}_{(q-1)/2}$-arrays, Szekeres $\mathbb{Z}_p^m$-array pairs, Paley $\mathbb{Z}_p^m$-array pairs, and Baumert $\mathbb{Z}_{p_1}^{m_1} \times \mathbb{Z}_{p_2}^{m_2}$-array pairs. Our work also solves one of the two open problems posed in Ding [J. Combin. Des. 16 (2008), 164-171]. Moreover, we provide several computer search based existence and non-existence results regarding Legendre $\mathbb{Z}_n$-array pairs. Finally, by using cyclotomic cosets, we provide a previously unknown Legendre $\mathbb{Z}_{57}$-array pair.

Keywords: Cyclotomy, Group ring, Hadamard matrix, Skew-symmetric, Supplementary difference set

1 Introduction

In this section, we first survey several known infinite binary G-array families and G-array pairs for a finite abelian group G. In Section 2, we show how these G-array families and G-array pairs are related to each other.

1.1 G-arrays and their correlations

Let n be a positive integer and G be an abelian group of order n. Then $\boldsymbol{a} = (a_g)$ with $g \in G$ and $a_g \in \mathbb{C}$ is called a *G-array*. The *cross-correlation function* of the two G-arrays (a_g) and (b_g) is defined by:

$$C_{\boldsymbol{a},\boldsymbol{b}}(t) = \sum_{g \in G} a_{gt} \bar{b}_g,$$

where $t \in G$ and $\bar{b}_g$ is the complex conjugate of b_g. If $\boldsymbol{a} = \boldsymbol{b}$, then $C_{\boldsymbol{a},\boldsymbol{a}}(t) := C_{\boldsymbol{a}}(t)$ is called the *autocorrelation function* of $\boldsymbol{a}$.

We call a G-array $\boldsymbol{a}$ a $\{0,1\}$ ($\{-1,1\}$) G-array if $a_g \in \{0,1\}$ ($\{-1,1\}$) $\forall g \in G$. In this paper, we consider only $\{-1,1\}$ or $\{0,1\}$ G-arrays. The linear transformation $a_g \to 2a_g - 1$ is a bijection that maps a $\{0,1\}$ G-array to a $\{-1,1\}$ G-array. Throughout, we switch repeatedly between a $\{0,1\}$ G-array and its corresponding $\{-1,1\}$ G-array. The choice between $\{0,1\}$ and $\{-1,1\}$ coefficients in any particular context is dictated by applications or ease of computation. If we refer to a $\{0,1\}$ G-array as a $\{-1,1\}$ G-array we mean the $\{-1,1\}$ G-array obtained from the $\{0,1\}$ G-array by applying the bijection $a_g \to 2a_g - 1$.

By the structure theorem, every finite abelian group G is isomorphic to $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$ for some $r \in \mathbb{Z}^{\geq 1}$. Let $H_i = \langle \omega_i \rangle$ and $|H_i| = s_i$ for $s_i \in \mathbb{Z}^{\geq 2}$. Then, the map $\Theta : \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r} \to H_1 \times \cdots \times H_r$ such that $\Theta(\alpha_1, \ldots, \alpha_r) = \omega_1^{\alpha_1} \ldots \omega_r^{\alpha_r}$ is an isomorphism between $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$ and $H_1 \times \cdots \times H_r$ for each set of fixed $\{\omega_i\}_{i=1}^r$. Throughout the paper we fix the notation Θ for this isomorphism.

For a G-array (a_g) and an isomorphism $\Phi : G \to \Phi(G)$, define the $\Phi(G)$-array $\Phi(a_g)$ via

$$\Phi((a_g)) = (a'_{\Phi(g)}), \text{ where } a'_{\Phi(g)} = a_g.$$

Clearly, both the autocorrelation and the cross-correlation functions are preserved under the map $g \to \Phi(g)$ for any isomorphism Φ, i.e. $C_{a,b}(t) = C_{\Phi(a),\Phi(b)}(\Phi(t))$ for any two G-arrays (a_g) and (b_g) where $g, t \in G$. Also, whenever we are using an isomorphic copy of G that has the form $H_1 \times \cdots \times H_r$, we say that G is *written multiplicatively*, and if G has the form $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$ we say that G is *written additively*. Unless otherwise specified, for a multiplicatively (additively) written group we use 1 (0) as the identity element. We also use e as the identity element of a group G.

Let $n = |G|$. Let $\boldsymbol{a} = (a_g)$ be a $\{-1,1\}$ or $\{0,1\}$ G-array. Then the set $D = \{g \mid g \in G \text{ and } a_g = 1\}$ is called the *set of* 1 *indices* of $\boldsymbol{a}$. Let $d_D(t) = |(Dt) \cap D|$, where Dt is the set of elements of D multiplied by t. Then $d_D(t)$ is called the *difference function* of $D \subseteq G$, and for a $\{0,1\}$ G-array $\boldsymbol{a}$ we have

$$C_{\boldsymbol{a}}(t) = d_D(t).$$

Hence, the autocorrelation function measures how much a $\{0,1\}$ G-array differs from its translates. When $\boldsymbol{a} = (a_g)$ is a $\{-1,1\}$ G-array we get

$$C_{\boldsymbol{a}}(t) = n - 4(k - d_D(t)), \tag{1}$$

where $k = |D|$, see [13]. By equation (1) if $\boldsymbol{a} = (a_g)$ is a $\{-1,1\}$ G-array, then

$$C_{\boldsymbol{a}}(t) \equiv n \pmod 4.$$

A $\{-1,1\}$ G-array $\boldsymbol{a}$ is called *perfect* if for $t \neq e$

$$C_{\boldsymbol{a}}(t) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4, \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ \pm 2 & \text{if } n \equiv 2 \pmod 4, \\ -1 & \text{otherwise.} \end{cases}$$

A $\{-1,1\}$ G-array $\boldsymbol{a} = (a_g)$ is called *balanced* if

$$\sum_{g \in G} a_g = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2, \\ \pm 1 & \text{otherwise,} \end{cases}$$

and *almost balanced* if

$$\sum_{g \in G} a_g = \begin{cases} \pm 2 & \text{if } n \equiv 0 \pmod 2, \\ \pm 3 & \text{otherwise.} \end{cases}$$

Then, based on equation (1), a $\{0,1\}$ G-array $\boldsymbol{a}$ with $\sum_{g \in G} a_g = k$ is defined to be *perfect* if for $t \neq e$

$$C_{\boldsymbol{a}}(t) = d_D(t) = \begin{cases} k - \frac{n}{4} & \text{if } n \equiv 0 \pmod 4, \\ k - \frac{n-1}{4} & \text{if } n \equiv 1 \pmod 4, \\ k - \frac{n \pm 2}{4} & \text{if } n \equiv 2 \pmod 4, \\ k - \frac{n+1}{4} & \text{otherwise,} \end{cases} \tag{2}$$

and a $\{0,1\}$ G-array $\boldsymbol{a} = (a_g)$ is defined to be *balanced* if

$$\sum_{g \in G} a_g = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2, \\ \frac{n \pm 1}{2} & \text{otherwise,} \end{cases} \tag{3}$$

and *almost balanced* if

$$\sum_{g \in G} a_g = \begin{cases} \frac{n \pm 2}{2} & \text{if } n \equiv 0 \pmod 2, \\ \frac{n \pm 3}{2} & \text{otherwise.} \end{cases} \tag{4}$$

A G-array $\boldsymbol{a}$ is said to have *good matched autocorrelation properties* if

$$\max_{t\in G\setminus\{e\}} |C_{\boldsymbol{a}}(t)|,$$

and

$$\sum_{t\in G} |C_{\boldsymbol{a}}(t)|^2$$

are both small, where $\max_{t\in G\setminus\{e\}} |C_{\boldsymbol{a}}(t)|$ is called the *peak correlation* and $\sum_{t\in G} |C_{\boldsymbol{a}}(t)|^2$ is called the *correlation energy*.

Let G be a group of order v and D be a subset of G with k elements. For any $\alpha \neq e$ and $\alpha \in G$ if the equation

$$d(d')^{-1} = \alpha \tag{5}$$

has exactly λ solution pairs (d, d') with both d and d' in D, then the set D is called a *difference set* in G with parameters (v, k, λ) denoted by $\mathrm{DS}(v, k, \lambda)$. If equation (5) has λ solutions for t of the non-identity elements of G and $\lambda + 1$ solutions for every other non-identity element, then D is called an *almost difference set* in G with parameters (v, k, λ, t) denoted by $\mathrm{ADS}(v, k, \lambda, t)$. If G is an abelian (cyclic) group and D is a difference set, then D is called an *abelian (cyclic) difference set* in G. If G is an abelian (cyclic) group and D is an almost difference set, then D is called an *abelian (cyclic) almost difference set*.

Clearly, D is a(n) (almost) difference set in G if and only if $\Phi(D)$ is a(n) (almost) difference set in $\Phi(G)$ for any isomorphism $\Phi : G \to \Phi(G)$. For a survey of almost difference sets, see [2].

A $\{-1, 1\}$ $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$-array with k entries equal to 1 and all nontrivial autocorrelation coefficients equal to $\theta = n - 4(k - \lambda)$ is equivalent to an abelian $\mathrm{DS}(n, k, \lambda)$, see Lemma 1.3 in [10].

Supplementary difference sets generalize the concept of difference sets [19].

Definition 1. Let G be a group of order v. A collection $D_1, D_2, \ldots, D_f$ of f subsets of G with $|D_i| = k_i$ is called a *supplementary difference set* in G denoted by f-$\mathrm{SDS}(v; k_1, \ldots, k_f; \lambda)$ if for each $\alpha \in G \setminus \{e\}$, the constraint

$$\alpha = xy^{-1},$$

where $x, y \in D_i$ for some $i \in \{1, 2, \ldots, f\}$, has exactly λ solutions.

Clearly, $D_1, \ldots, D_f$ is a f-$\mathrm{SDS}(v; k_1, \ldots, k_f; \lambda)$ in G if and only if $\Phi(D_1), \ldots, \Phi(D_f)$ is an f-$\mathrm{SDS}(v; k_1, \ldots, k_f; \lambda)$ in $\Phi(G)$ for any isomorphism $\Phi : G \to \Phi(G)$.

1.2 The group ring notation

First, we introduce the group ring notation that will be used in the proofs.

Definition 2. Let G be a multiplicatively written finite abelian group and R be a ring. Then, the *group ring* of G over R is the set denoted by $R[G]$ defined as:

$$R[G] = \left\{ \sum_{g \in G} a_g g \mid a_g \in R \right\}.$$

$R[G]$ is a free R-module of rank $|G|$. Any group isomorphism $\Phi : G \to \Phi(G)$ extends linearly to a module and group ring isomorphism between $R[G]$ and $R[\Phi(G)]$, where

$$\Phi(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g \Phi(g) = \sum_{\Phi(g) \in \Phi(G)} a_{\Phi(g)} \Phi(g).$$

If G is a multiplicatively written group, then multiplication and addition in $R[G]$ are defined in the same way as in the ring of formal Laurent series $R[[x_1, \dots x_n]]$. If G is additively written, then there exists an isomorphism $\Phi : G \to \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$ for some $s_1, \dots, s_r$. In this case, addition in $R[G]$ is defined the same way as in the case when G is multiplicatively written. The multiplication of two elements $u, v \in R[G]$ is defined as

$$u * v = (\Theta\Phi)^{-1}(\Theta\Phi(u)\Theta\Phi(v)).$$

For short hand notation, we define the *power* of a group ring element in the following way.

Definition 3. If $W = \sum_{g \in G} a_g g$ is an element of $R[G]$ and t some integer, then

$$W^{(t)} = \sum_{g \in G} a_g g^t, \quad \overline{W} = \sum_{g \in G} \bar{a}_g g, \quad \text{and} \quad |W| = \sum_{g \in G} |a_g|.$$

The following are two remarks concerning Definition 3.

1. For a group ring element A in this paper we always have $\overline{A} = A$.
2. The element $\left(\sum_{g \in G} a_g g\right)^{(t)}$ is not the same as the element $\left(\sum_{g \in G} a_g g\right)^t$.

Let $D \subseteq G$ with $|D| = k$ and $A = \sum_{g \in D} g$. Then, D is a DS(v, k, λ) if and only if

$$AA^{(-1)} = (k - \lambda)(1) + \lambda \left(\sum_{g \in G} g \right) \in \mathbb{Z}[G].$$

We can think of a G-array as a matrix. Let $\boldsymbol{M}$ be a matrix whose rows and columns are indexed by the elements in G. Define

$$P = \{g \,|\, m_{1,g} = +1\},$$

and

$$N = \{g \,|\, m_{1,g} = -1\}.$$

Let the G-array $m_{1,g}$ be the first row of $\boldsymbol{M}$. Then, the remaining rows of $\boldsymbol{M}$ can be obtained by setting

$$m_{g,h} = \begin{cases} 1, & \text{if } gh^{-1} \in P, \\ -1, & \text{if } gh^{-1} \in N. \end{cases}$$

A matrix developed this way is called *G-developed* or *G-circulant*.

For a cyclic group G, if the rows and columns of a matrix $\boldsymbol{M}$ are indexed by successive powers of a generator of G, then the G-developed matrix $\boldsymbol{M}$ is called *circulant*. Alternatively, a circulant matrix $\boldsymbol{A} = circ(\boldsymbol{a})$ is determined by its first column, where each column (row) of $\boldsymbol{A}$ is a cyclic down (right) shift of the vector $\boldsymbol{a}$. An $m_1 m_2 \times m_1 m_2$ matrix $\boldsymbol{C}$ is said to be *block-circulant* if it is of the form

$$\boldsymbol{C} = circ(\boldsymbol{C}_0, \boldsymbol{C}_1, \ldots, \boldsymbol{C}_{m_2-1}) = \begin{bmatrix} \boldsymbol{C}_0 & \boldsymbol{C}_{m_2-1} & \cdots & \boldsymbol{C}_1 \\ \boldsymbol{C}_1 & \boldsymbol{C}_0 & \cdots & \boldsymbol{C}_2 \\ \boldsymbol{C}_2 & \boldsymbol{C}_1 & \cdots & \boldsymbol{C}_3 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}_{m_2-1} & \boldsymbol{C}_{m_2-2} & \cdots & \boldsymbol{C}_0 \end{bmatrix}, \tag{6}$$

where the $\boldsymbol{C}_j$ are $m_1 \times m_1$ matrices. If each $\boldsymbol{C}_i$ in equation (6) is itself also circulant then $\boldsymbol{C}$ is a block-circulant matrix of circulant matrices. More generally, if the group G is abelian but not cyclic then $G \cong \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$ for some $r \geq 2$ and the G-developed matrix is *r-circulant* that is obtained after applying the $circ(.)$ operator r times.

For a permutation Π of indices in $\{1, \ldots, n\}$, let $\boldsymbol{P}_\Pi$ be the corresponding $n \times n$ permutation matrix. Then, the *automorphism group* $\mathrm{Aut}(\boldsymbol{A})$ of an $n \times n$ matrix $\boldsymbol{A}$ is defined to be

$$\mathrm{Aut}(\boldsymbol{A}) = \{\Pi \,|\, \boldsymbol{P}_\Pi \boldsymbol{A} \boldsymbol{P}_\Pi^\top = \boldsymbol{A}\}.$$

For a G-developed matrix $\boldsymbol{A}$, if we permute the indices of $\boldsymbol{A}$ by the action of multiplication by elements of G, then the elements of G can be thought of as a set of permutations matrices that form a subgroup of $\mathrm{Aut}(\boldsymbol{A})$. Hence, $\mathrm{Aut}(\boldsymbol{A}) \geq G$ and it is easy to construct examples where $\mathrm{Aut}(\boldsymbol{A}) > G$. The set of all matrices whose automorphism group contains G and entries are in R is isomorphic to $R[G]$. This follows by taking $X = G$ on page 4 in [9]. In particular, the products and integer linear combinations of circulant (r-circulant) matrices is circulant (r-circulant).

There is an injection Ψ of $\{0,1\}$ or $\{-1,1\}$ G-arrays into $\mathbb{Z}[G]$ given by

$$\Psi(\boldsymbol{a}) = \sum_{g\in G} a_g g.$$

We say that the G-array $\boldsymbol{a}$ *corresponds* to $A \in \mathbb{Z}[G]$ if $A = \sum_{g\in G} a_g g$. For a group ring element $\sum_{g\in G} a_g g$ corresponding to a G-array (a_g) and an isomorphism $\Phi : G \to \Phi(G)$ we define $\Phi(A)$ to be

$$\Phi(A) = \sum_{g\in G} a_g \Phi(g) = \sum_{\Phi(g)\in\Phi(G)} a_g \Phi(g).$$

Throughout the paper, by abuse of notation, if a set $H \subseteq G$ appears in a group ring equation, it is understood that $H = \sum_{h\in H} h$. Moreover, for $A = \sum_{g\in G} a_g g \in \mathbb{Z}[G]$, we define

$$\{A\} := \{g \in G \mid a_g = 1\}.$$

The group ring elements that correspond to G-arrays are used to calculate the autocorrelation and cross-correlation functions of G-arrays, i.e., for a multiplicatively written group G, and G-arrays $\boldsymbol{a}$ and $\boldsymbol{b}$

$$C_{\boldsymbol{a},\boldsymbol{b}}(t) = \text{coefficient of } t \text{ in } A\overline{B}^{(-1)}, \tag{7}$$

where $A = \sum_{g\in G} a_g g$, $B = \sum_{g\in G} b_g g$.

A matrix $\boldsymbol{M}$ with entries in $\mathbb{R}$ is *symmetric (skew-symmetric)* if $\boldsymbol{M} = \boldsymbol{M}^\top$ ($\boldsymbol{M} = -\boldsymbol{M}^\top$). Next, we define symmetric, skew-symmetric G-arrays, and skew-type matrices.

Definition 4. Let G be a finite group with identity e. Let $\boldsymbol{m} = (m_g)$ be a $\{0,1\}$ G-array and $M = \sum_{g\in G} m_g g$. Then, $\boldsymbol{m}$ or M is *symmetric* if $M = M^{(-1)}$ and *skew-symmetric* if $M + M^{(-1)} = G + e$ (implying $e \in \{M\}$) or $M + M^{(-1)} = G - e$ (implying $e \notin \{M\}$).

The following lemma shows that an isomorphism $\Phi : G \to \Phi(G)$ maps a symmetric (skew-symmetric) G-array to a symmetric (skew-symmetric) $\Phi(G)$-array.

Lemma 1. Let (a_g) be a symmetric (skew-symmetric) G-array. Let Φ be an isomorphism $\Phi : G \to \Phi(G)$. Then $\Phi((a_g))$ is a symmetric (skew-symmetric) G-array.

Proof. Let $A = \sum_{g\in G} a_g g \in \mathbb{Z}[G]$ and Φ be extended linearly to an isomorphism of $R[G]$ and $R[\Phi(G)]$. Then, $A = A^{(-1)}$ implies $\Phi(A) = \Phi(A^{(-1)})$ ($M + M^{(-1)} = G + e$ implies $\Phi(M) + \Phi(M^{(-1)}) = \Phi(G) + \Phi(e)$ and $M + M^{(-1)} = G - e$ implies $\Phi(M) + \Phi(M^{(-1)}) = \Phi(G) - \Phi(e)$). □

The following lemma follows immediately from Definition 4.

Lemma 2. Let G be a finite group with identity e and M be the group ring element corresponding to $\boldsymbol{m}$. Then, a $\{0,1\}$ G-array $\boldsymbol{m}$ is symmetric (skew-symmetric) if and only if $\{M\} = \{M^{(-1)}\}$ ($\{M\} \cup \{M^{(-1)}\} = G \backslash e$, $\{M\} \cap \{M^{(-1)}\} = \emptyset$ when $e \notin \{M\}$ and $\{M\} \cup \{M^{(-1)}\} = G$, $\{M\} \cap \{M^{(-1)}\} = e$ when $e \in \{M\}$).

A matrix $\boldsymbol{M}$ is of *skew-type* if $\text{Diag}(\boldsymbol{M}) = \boldsymbol{I}$ and $(\boldsymbol{M} - \text{Diag}(\boldsymbol{M}))$ is skew-symmetric, where $\text{Diag}(\boldsymbol{M})$ is the diagonal matrix obtained from $\boldsymbol{M}$ by replacing each non-diagonal entry of $\boldsymbol{M}$ with 0. Now, it is plain to see the following lemma.

Lemma 3. Let $\boldsymbol{m} = (m_g)$ be a $\{0,1\}$ G-array with $m_e = 1$. Let $\boldsymbol{M}$ be the group developed matrix obtained by using $m_{e,g} = m_g$ as its first row and $\boldsymbol{J}_{|G|\times|G|}$ be the $|G| \times |G|$ matrix of all 1s. Then $2\boldsymbol{M} - \boldsymbol{J}_{|G|\times|G|}$ is symmetric (skew-type) if and only if $M = \sum_{g\in G} m_g g$ is symmetric (skew-symmetric).

1.3 Legendre G-array pairs

First, we define Legendre G-array pairs.

Definition 5. Let G be a multiplicatively written finite abelian group with $|G| = n$. Then, a pair of $\{-1,1\}$ G-arrays $(\boldsymbol{a} = (a_g), \boldsymbol{b} = (b_g))$ form a *Legendre G-array pair* if $\sum_{g\in G} a_g = \sum_{g\in G} b_g$ and

$$AA^{(-1)} + BB^{(-1)} = (|A| + |B|)\,(1) - 2(G-1), \tag{8}$$

where A and B are the group ring elements associated with $\boldsymbol{a}$ and $\boldsymbol{b}$.

By applying the principal character to the group ring equation (8) we get

$$\begin{aligned} \chi_0\left(AA^{(-1)} + BB^{(-1)}\right) &= \chi_0\left((|A| + |B|)(1) - 2(G-1)\right) \\ \chi_0(A)^2 + \chi_0(B)^2 &= 2n - 2(n-1) \\ a^2 + b^2 &= 2. \end{aligned} \tag{9}$$

This equation implies that $a = b \in \{-1,1\}$, where $a = \sum_{g\in G} a_g = b = \sum_{g\in G} b_g$. Thus $|G| = n$ must be odd for a Legendre G-array pair to exist. Hence, each G-array in a Legendre G-array pair must be balanced.

By equations (1), (8), and (9) we get the following definition of Legendre $\{0,1\}$ G-array pairs.

Definition 6. Let G be a multiplicatively written finite abelian group. A pair of $\{0,1\}$ G-arrays $(\boldsymbol{a} = (a_g), \boldsymbol{b} = (b_g))$ form a *Legendre G-array pair* if $\sum_{g\in G} a_g = \sum_{g\in G} b_g$, and

$$AA^{(-1)} + BB^{(-1)} = \begin{cases} 2\left(\frac{|G|+1}{2}\right)(1) + \frac{|G|+1}{2}(G-1) & \text{if } \sum_{g\in G} a_g = \sum_{g\in G} b_g = \frac{|G|+1}{2}, \\ 2\left(\frac{|G|-1}{2}\right)(1) + \frac{|G|-3}{2}(G-1) & \text{if } \sum_{g\in G} a_g = \sum_{g\in G} b_g = \frac{|G|-1}{2}. \end{cases}$$

The following lemma is plain to prove.

Lemma 4. Let $\Phi : G \to \Phi(G)$ be an isomorphism. Then, $((a_g),(b_g))$ is a Legendre $\{-1,1\}$ ($\{0,1\}$) G-array pair if and only if $(\Phi((a_g)),\Phi((b_g)))$ is a Legendre $\{-1,1\}$ ($\{0,1\}$) $\Phi(G)$-array pair. Hence, whenever we construct a Legendre $\{-1,1\}$ ($\{0,1\}$) G-array pair we have also constructed a Legendre $\{-1,1\}$ ($\{0,1\}$) $\Phi(G)$-array pair.

The following well-known theorem connects supplementary difference sets in finite abelian groups and Legendre G-array pairs.

Theorem 1. Let G be an abelian group of order n. Let $(\boldsymbol{a},\boldsymbol{b})$ be a $\{0,1\}$ or $\{-1,1\}$ G-array pair and (M,N) be the subsets of G such that $M = \{g \in G \mid a_g = 1\}$ and $N = \{g \in G \mid b_g = 1\}$. Then, (M,N) is a 2-SDS$(n;(n+1)/2,(n+1)/2;(n+1)/2)$ or a 2-SDS$(n;(n-1)/2,(n-1)/2;(n-3)/2)$ if and only if $(\boldsymbol{a},\boldsymbol{b})$ is a Legendre G-array pair.

Proof. If G is written multiplicatively, then the result follows by comparing Definition 1 for $f=2$ to Definition 5 (Definition 6) for $\{-1,1\}$ ($\{0,1\}$) G-arrays. □

It is conjectured that a Legendre $\mathbb{Z}_n$-array pair exists for all odd n [8]. A Legendre $\mathbb{Z}_n$-array pair is known to exist when:

- n is a prime, see [8];
- $2n+1$ is a prime power (Szekeres, [20]);
- $n = 2^m - 1$ for $m \geq 2$, see [15];
- $n = p_1(p_1+2)$, with $p_2 = p_1 + 2$, where p_1, p_2 are odd primes [4].

Currently, $n = 77$ is the smallest n for which no Legendre $\mathbb{Z}_n$-array pair is known.

An $N \times N$ *Hadamard matrix*, $\boldsymbol{H}$, is a ± 1 matrix such that $\boldsymbol{H}\boldsymbol{H}^\top = N\boldsymbol{I}_N$ where $\boldsymbol{I}_N$ is the identity matrix of order N. The following theorem showing that the existence of a Legendre G-array pair implies the existence of a $(2|G|+2)\times(2|G|+2)$ Hadamard matrix is well-known.

Theorem 2. Let $(\boldsymbol{a},\boldsymbol{b})$ be a Legendre $\{-1,1\}$ G-array pair, with $|G| = n$ such that $\sum_{g\in G} a_g = \sum_{g\in G} b_g = 1$. Let $(\boldsymbol{a},\boldsymbol{b})$ be developed into G indexed $n \times n$ matrices $\boldsymbol{A}$ and

$\boldsymbol{B}$ by taking $\boldsymbol{a}$ and $\boldsymbol{b}$ as the first row of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. Let

$$\boldsymbol{H}_{sym} = \begin{bmatrix} - & - & + & \cdots & + & + & \cdots & + \\ - & + & + & \cdots & + & - & \cdots & - \\ + & + & & & & & & \\ \vdots & \vdots & & \boldsymbol{A} & & & \boldsymbol{B} & \\ + & + & & & & & & \\ + & - & & & & & & \\ \vdots & \vdots & & \boldsymbol{B}^\top & & & -\boldsymbol{A}^\top & \\ + & - & & & & & & \end{bmatrix}$$

and

$$\boldsymbol{H}_{skew} = \begin{bmatrix} + & + & + & \cdots & + & + & \cdots & + \\ - & + & + & \cdots & + & - & \cdots & - \\ - & - & & & & & & \\ \vdots & \vdots & & \boldsymbol{A} & & & \boldsymbol{B} & \\ - & - & & & & & & \\ - & + & & & & & & \\ \vdots & \vdots & & -\boldsymbol{B}^\top & & & \boldsymbol{A}^\top & \\ - & + & & & & & & \end{bmatrix}.$$

Then, both $\boldsymbol{H}_{sym}$ and $\boldsymbol{H}_{skew}$ are Hadamard matrices. Moreover, $\boldsymbol{H}_{sym}$ ($\boldsymbol{H}_{skew}$) is symmetric (skew-type) Hadamard matrix if and only if $\boldsymbol{a}$ is symmetric (skew-symmetric).

Proof. Let e be the identity element in G. The matrix $\boldsymbol{H}_{sym}$ ($\boldsymbol{H}_{skew}$) is a Hadamard matrix if and only if $C_{\boldsymbol{a}}(t) + C_{\boldsymbol{b}}(t) = -2$ for all $t \in G \setminus \{e\}$. Then, by using equation (7) for $C_{\boldsymbol{a},\boldsymbol{a}}$ and $C_{\boldsymbol{b},\boldsymbol{b}}$, we get $C_{\boldsymbol{a}}(t) + C_{\boldsymbol{b}}(t) = -2$ for all $t \in G \setminus \{e\}$ if and only if $(\boldsymbol{a}, \boldsymbol{b})$ is a Legendre G-array pair. The matrix $\boldsymbol{H}_{sym}$ ($\boldsymbol{H}_{skew}$) is symmetric (skew-type) if and only if $\boldsymbol{A}$ is symmetric (skew-type). The result now follows as $\boldsymbol{A}$ is symmetric (skew-type) if and only if $\boldsymbol{a}$ is symmetric (skew-symmetric). $\square$

Consider the action of the group $(\mathbb{Z}_{s_1} \rtimes \mathbb{Z}_{s_1}^*) \times \cdots \times (\mathbb{Z}_{s_r} \rtimes \mathbb{Z}_{s_r}^*)$ on the group $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$ defined by

$$((a_1, b_1), \ldots, (a_r, b_r))(g_1, \ldots, g_r) = (b_1 g_1 + a_1, \ldots, b_r g_r + a_r)$$

if the group $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$ is written additively, and

$$((a_1, b_1), \ldots, (a_r, b_r))(g_1, \ldots, g_r) = (g_1^{b_1} a_1, \ldots, g_r^{b_r} a_r)$$

if $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$ is written multiplicatively, where $\mathbb{Z}_{s_i}^*$ is the multiplicative group of the ring $\mathbb{Z}_{s_i}$ and $\rtimes$ is the *semidirect product* as defined in [14, p. 167]. This group action can be extended linearly to $\mathbb{Z}[G]$. Then, $(\mathbb{Z}_{s_1} \rtimes \mathbb{Z}_{s_1}^*) \times \cdots \times (\mathbb{Z}_{s_r} \rtimes \mathbb{Z}_{s_r}^*)$ acts on a $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$-array, and two $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$-arrays are called *equivalent* if one can be obtained from the other by applying the elements of the group

$$(\mathbb{Z}_{s_1} \rtimes \mathbb{Z}_{s_1}^*) \times \cdots \times (\mathbb{Z}_{s_r} \rtimes \mathbb{Z}_{s_r}^*).$$

It is well-known that if $\boldsymbol{a}$ and $\boldsymbol{a}'$ are equivalent $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$-arrays then $\boldsymbol{a}$ and $\boldsymbol{a}'$ have the same peak correlation and the same correlation energy. We call two Legendre pairs $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{a}', \boldsymbol{b}')$ *equivalent* if $\{\boldsymbol{a}, \boldsymbol{b}\} = \{\tau \boldsymbol{a}', \beta \boldsymbol{b}'\}$, where $\tau = ((\tau_1, \tau_1^*), \ldots, (\tau_r, \tau_r^*))$, $\beta = ((\beta_1, \beta_1^*), \ldots, (\beta_r, \beta_r^*))$ such that $\beta_i, \tau_i \in \mathbb{Z}_{s_i}$, $\beta_i^*, \tau_i^* \in \mathbb{Z}_{s_i}^*$ and $\tau_i^* = \pm \beta_i^*$ for $i = 1, \ldots, r$ [8]. If $(\boldsymbol{a}, \boldsymbol{b})$ is a Legendre $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$-array pair, and $(\boldsymbol{a}', \boldsymbol{b}')$ is equivalent to $(\boldsymbol{a}, \boldsymbol{b})$, then $(\boldsymbol{a}', \boldsymbol{b}')$ is also a Legendre $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$-array pair.

The following lemma determines exactly which subgroup of $(\mathbb{Z}_{s_1} \rtimes \mathbb{Z}_{s_1}^*) \times \cdots \times (\mathbb{Z}_{s_r} \rtimes \mathbb{Z}_{s_r}^*)$ preserves symmetry (skew-symmetry) of a symmetric (skew-symmetric) $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$-array.

Lemma 5. The group $(\{0\} \rtimes \mathbb{Z}_{s_1}^*) \times \cdots \times (\{0\} \rtimes \mathbb{Z}_{s_r}^*)$ preserves the symmetry (skew-symmetry) of a symmetric (skew-symmetric) $\{-1, 1\}$ or $\{0, 1\}$ $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$-array.

Proof. Let $\boldsymbol{a}$ be a symmetric (skew-symmetric) $\{-1, 1\}$ or $\{0, 1\}$ $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$-array, and $A \subset \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$ be the set of 1 indices of $\boldsymbol{a}$. Then, $A = -A$ (($A \cup -A = \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r} \backslash \{0\}$ and $A \cap -A = \emptyset$) or ($A \cup -A = \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$ and $A \cap -A = 0$)) implies for any $((0, \beta_1^*), \ldots, (0, \beta_r^*)) \in (\{0\} \rtimes \mathbb{Z}_{s_1}^*) \times \cdots \times (\{0\} \rtimes \mathbb{Z}_{s_r}^*)$

$$((0, \beta_1^*), \ldots, (0, \beta_r^*))A = -((0, \beta_1^*), \ldots, (0, \beta_r^*))A$$

$$\begin{aligned} (((0, \beta_1^*), \ldots, (0, \beta_r^*))A \cap -((0, \beta_1^*), \ldots, (0, \beta_r^*))A &= \emptyset \quad \text{or} \\ ((0, \beta_1^*), \ldots, (0, \beta_r^*))A \cap -((0, \beta_1^*), \ldots, (0, \beta_r^*))A &= 0). \end{aligned}$$

□

In general, Lemma 5 can not be improved as it is easy to construct a symmetric (skew-symmetric) $\mathbb{Z}_s$-array whose symmetry (skew-symmetry) is not preserved by any circulant shifts other than the 0 shift.

We fix some notation that will be used in the rest of the paper. Let $q = p^m$ for some prime p and positive integer m. Let $\mathbb{F}_q$ be the finite field with q elements and $\mathbb{F}_q^* = \langle \alpha \rangle$ be the multiplicative group of $\mathbb{F}_q$, where α is a generator for $\mathbb{F}_q^*$. Let $C_0^{(d,q,\alpha)} = \langle \alpha^d \rangle$ be the multiplicative group generated by α^d in the finite field $\mathbb{F}_q$, where d divides $q-1$. Observe that $C_0^{(d,q,\alpha)}$ does not depend on α. Let $C_i^{(d,q,\alpha)} = \alpha^i C_0^{(d,q,\alpha)}$ for $i = 0, 1, \ldots, d-1$, where $C_i^{(d,q,\alpha)}$ are called *cyclotomic classes of order* d, see [17]. We will denote $C_i^{(d,q,\alpha)}$ with C_i^d when there is no need to specify q and α. The labeling of $C_1^{(d,q,\alpha)}, \ldots, C_{d-1}^{(d,q,\alpha)}$ depends on α, but taking a different choice of primitive root just permutes $C_1^{(d,q,\alpha)}, \ldots, C_{d-1}^{(d,q,\alpha)}$.

1.4 Infinite families of perfect G-arrays

First, we survey several known infinite families of perfect G-arrays.
The Sidelnikov-Lempel-Cohn-Eastman $\mathbb{Z}_{q-1}$-arrays:
Let

$$S = \{\alpha^{2i+1} - 1\}_{i=0}^{\frac{q-1}{2}-1}.$$

Let $\boldsymbol{a}$ be a $\{-1, 1\}$ or $\{0, 1\}$ $(q-1) \times 1$ vector such that

$$a_i = 1 \quad \text{if } \alpha^i \in S.$$

Then, $\boldsymbol{a}$ is the Sidelnikov-Lempel-Cohn-Eastman $\mathbb{Z}_{q-1}$-array. The Sidelnikov-Lempel-Cohn-Eastman $\mathbb{Z}_{q-1}$-array is always balanced. However, it is perfect if and only if $q = p^m \equiv 3 \pmod 4$, see [12] and [16].
The Ding-Helleseth-Martinsen $\mathbb{Z}_2 \times \mathbb{Z}_p^m$-arrays:
Let $p \equiv 1 \pmod 4$ and $p^m = s^2 + 4t^2$, where $s^2 = 1$ or $t^2 = 1$. Let $q = p^m \equiv 5 \pmod 8$, or equivalently, let $p \equiv 5 \pmod 8$ and m be odd. Let $C_{i,j,\ell} = \left(C_i^4 \cup C_j^4, C_j^4 \cup C_\ell^4\right)$ for $\{i, j, l\} \subset \{0, 1, 2, 3\}$, where i, j, l are distinct integers. Let

$$(A_1, B_1) = (C_0^4 \cup C_1^4, C_1^4 \cup C_3^4), \quad (A_2, B_2) = (C_0^4 \cup C_2^4, C_2^4 \cup C_3^4) \quad \text{if } t^2 = 1,$$
$$(A_3, B_3) = (C_0^4 \cup C_1^4, C_0^4 \cup C_3^4) \quad \text{if } s^2 = 1.$$

Identify the elements of the finite field $\mathbb{F}_{p^m}$ with its additive group $\mathbb{Z}_p^m$, and let $\langle\omega\rangle = \Theta(\mathbb{Z}_2)$, where Θ be the isomorphism in Section 1.1. We now use the group $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)$ as an indexing set. For each $i \in \{1, 2, 3\}$, let the equivalence class i Ding-Helleseth-Martinsen $\{-1, 1\}$ or $\{0, 1\}$ $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)$-array be such that $\Theta(A_i) \cup \Theta(B_i)\omega$ is the set of 1 indices of the array. Then, each equivalence class of Ding-Helleseth-Martinsen $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)$-array is almost balanced, and equivalence class 3 is always perfect, see Theorem 2 in [7]. In Section 2.2 we determine exactly when each of the equivalence class 1 and 2 Ding-Helleseth-Martinsen $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)$-array is perfect. This solves one of the two open problems posed in [6]. Finally, each equivalence class of $\{-1, 1\}$ or $\{0, 1\}$ Ding-Helleseth-Martinsen $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)$-array (a_g) is used to construct the corresponding $\{-1, 1\}$ or $\{0, 1\}$ Ding-Helleseth-Martinsen $\mathbb{Z}_2 \times \mathbb{Z}_p^m$-array as $\Theta^{-1}((a_g))$.

1.5 Infinite families of Legendre G-array pairs

Now, we survey several known infinite families of Legendre G-array pairs.
The Yamada $\mathbb{Z}_{(q-1)/2}$-array pairs:
Let $q = p^m \equiv 3 \pmod 4$. Let

$$M = \{a : \alpha^{2a} + 1 \in C_0^2\},$$

and

$$N = \{a : \alpha^{2a} - 1 \in C_0^2\}.$$

Then the pair (M, N) is a 2-SDS$((q-1)/2; (q-3)/4, (q-3)/4; (q-7)/4)$ in $\mathbb{Z}_{(q-1)/2}$. Let $\mathbb{Z}_{(q-1)/2}$ index the arrays $\boldsymbol{a}, \boldsymbol{b}$, and (M, N) be the sets of 1 indices of $(\boldsymbol{a}, \boldsymbol{b})$. Then the $\{-1, 1\}$ or $\{0, 1\}$ $\mathbb{Z}_{(q-1)/2}$-array pair $(\boldsymbol{a}, \boldsymbol{b})$ is called a *Yamada* $\mathbb{Z}_{(q-1)/2}$*-array pair*, see [22]. The $\mathbb{Z}_{(q-1)/2}$-array $\boldsymbol{a}$ is symmetric and $\boldsymbol{b}$ is skew-symmetric.

The Szekeres $\mathbb{Z}_p^m$-array pairs:
Let $q = p^m \equiv 5 \pmod 8$, or equivalently, let $p \equiv 5 \pmod 8$ and m be odd. Let

$$A = C_0^4 \cup C_1^4, \ B = C_0^4 \cup C_3^4.$$

Then the pair (A, B) is a 2-SDS$(p^m; (q-1)/2, (q-1)/2; (q-3)/2)$ in $\mathbb{Z}_p^m$. Let $\mathbb{Z}_p^m$ index the arrays $\boldsymbol{a}, \boldsymbol{b}$, and (A, B) be the sets of 1 indices of $(\boldsymbol{a}, \boldsymbol{b})$. Then the $\{-1, 1\}$ or $\{0, 1\}$ $\mathbb{Z}_p^m$-array pair $(\boldsymbol{a}, \boldsymbol{b})$ is called a *Szekeres* $\mathbb{Z}_p^m$*-array pair*, see [18]. Both $\boldsymbol{a}$ and $\boldsymbol{b}$ are skew-symmetric.

The Szekeres-Whiteman $\mathbb{Z}_p^m$-array pairs:
Let $q = p^m$, $p \equiv 5 \pmod 8$ and m be even with $m \geq 2$. Let

$$A = C_0^8 \cup C_1^8 \cup C_2^8 \cup C_3^8, \ B = C_0^8 \cup C_1^8 \cup C_6^8 \cup C_7^8.$$

Then the pair (A, B) is a 2-SDS$(p^m; (q-1)/2, (q-1)/2; (q-3)/2)$ in $\mathbb{Z}_p^m$. Let $\mathbb{Z}_p^m$ index the arrays $\boldsymbol{a}, \boldsymbol{b}$, and (A, B) be the sets of 1 indices of $(\boldsymbol{a}, \boldsymbol{b})$. Then the $\{-1, 1\}$ or $\{0, 1\}$ $\mathbb{Z}_p^m$-array pair $(\boldsymbol{a}, \boldsymbol{b})$ is called a *Szekeres-Whiteman* $\mathbb{Z}_p^m$*-array pair*. Szekeres [18] proved that a Szekeres-Whiteman $\mathbb{Z}_p^m$-array pair is a Legendre $\mathbb{Z}_p^m$-array pair, while Whiteman [21] independently showed this result however only for $m \equiv 2 \pmod 4$. It is easy to see that both $\boldsymbol{a}$ and $\boldsymbol{b}$ are skew-symmetric.

The Paley $\mathbb{Z}_p^m$-array pairs:
Let

$$\begin{aligned} A = C_0^2, \ B = C_0^2 \quad & \text{if } p^m \equiv 3 \pmod 4, \\ A = C_1^2, \ B = C_0^2 \quad & \text{if } p^m \equiv 1 \pmod 4. \end{aligned}$$

Then the pair (A, B) is a 2-SDS$(p^m; (q-1)/2, (q-1)/2; (q-3)/2)$ in $\mathbb{Z}_p^m$, see [8]. Let $\mathbb{Z}_p^m$ index the arrays $\boldsymbol{a}, \boldsymbol{b}$, and (A, B) be the sets of 1 indices of $(\boldsymbol{a}, \boldsymbol{b})$. Then the $\{-1, 1\}$ or $\{0, 1\}$ $\mathbb{Z}_p^m$-array pair $(\boldsymbol{a}, \boldsymbol{b})$ is called a *Paley* $\mathbb{Z}_p^m$*-array pair*. Both $\boldsymbol{a}$ and $\boldsymbol{b}$ are skew-symmetric if $p^m \equiv 3 \pmod 4$ and symmetric otherwise.

The Baumert $\mathbb{Z}_{p_1}^{m_1} \times \mathbb{Z}_{p_2}^{m_2}$-array pairs:
Let $p_1^{m_1} + 2 = p_2^{m_2}$, where p_1, p_2 are odd primes and m_1, m_2 are positive integers. Let $q_1 = p_1^{m_1}$, $q_2 = p_2^{m_2}$, and

$$A = \left(C_0^{(2,q_1)} \times C_0^{(2,q_2)}\right) \bigcup \left(C_1^{(2,q_1)} \times C_1^{(2,q_2)}\right) \bigcup \left(\mathbb{F}_{q_1} \times \{0\}\right), \ \ B = A.$$

Since A is a $\mathrm{DS}(q_1(q_1+2),(q_1^2+2q_1-1)/2,(q_1-1)(q_1+3)/4)$ in $\mathbb{Z}_{p_1}^{m_1}\times\mathbb{Z}_{p_2}^{m_2}$ [4], the pair (A,B) is a 2-$\mathrm{SDS}(p_1^{m_1}p_2^{m_2};(q_1^2+2q_1-1)/2,(q_1^2+2q_1-1)/2;(q_1-1)(q_1+3)/2)$ in $\mathbb{Z}_{p_1}^{m_1}\times\mathbb{Z}_{p_2}^{m_2}$. Let $\mathbb{Z}_{p_1}^{m_1}\times\mathbb{Z}_{p_2}^{m_2}$ index the arrays $\boldsymbol{a},\boldsymbol{b}$, and (A,B) be the sets of 1 indices of $(\boldsymbol{a},\boldsymbol{b})$. Then the $\{-1,1\}$ or $\{0,1\}$ $\mathbb{Z}_{p_1}^{m_1}\times\mathbb{Z}_{p_2}^{m_2}$- array pair $(\boldsymbol{a},\boldsymbol{b})$ is called a *Baumert* $\mathbb{Z}_{p_1}^{m_1}\times\mathbb{Z}_{p_2}^{m_2}$*-array pair.* Both $\boldsymbol{a}$ and $\boldsymbol{b}$ are neither symmetric nor skew-symmetric.

2 Results

2.1 Yamada-Pott G-array pairs

Yamada-Pott G-array pairs first appeared in [22] and later in [13]. A Yamada-Pott $\{0,1\}$ G-array pair is a Legendre $\{0,1\}$ G-array pair with the added properties that one G-array is symmetric and the other is skew-symmetric. In group ring notation we have the following definition.

Definition 7. Let G be a finite abelian group written multiplicatively. A Legendre $\{0,1\}$ G-array pair $(\boldsymbol{a},\boldsymbol{b})$ with $A=\sum_{g\in G}a_g g$ and $B=\sum_{g\in G}b_g g$ is a *Yamada-Pott* $\{0,1\}$ *G-array pair* if $|A|=|B|$ and:

1. $A=A^{(-1)}$;

2. $B+B^{(-1)}=G+1$ (implying $1\in\{B\}$) or $B+B^{(-1)}=G-1$ (implying $1\notin\{B\}$)

are satisfied.

The following lemma is plain to prove.

Lemma 6. Let $\Phi:G\to\Phi(G)$ be an isomorphism. Then, $((a_g),(b_g))$ is a Yamada-Pott $\{0,1\}$ G-array pair if and only if $(\Phi((a_g)),\Phi((b_g)))$ is a Yamada-Pott $\{0,1\}$ $\Phi(G)$-array pair.

By Lemma 6, whenever we construct a Yamada-Pott $\{0,1\}$ G-array pair, we have also constructed a Yamada-Pott $\{0,1\}$ $\Phi(G)$-array pair. The following theorem implies that the existence of a Yamada-Pott $\{0,1\}$ $\mathbb{Z}_u$-array pair implies the existence of a perfect $\{0,1\}$ $\mathbb{Z}_{2u}$-array.

Theorem 3. Let H be an abelian group with $|H|=u$ written multiplicatively and (A,B) be a Yamada-Pott $\{0,1\}$ H-array pair. Let $S=A+\omega B$ and $G=\langle\omega\rangle H$, where $\omega^2=1$, $\omega\neq 1$, and $\omega h=h\omega$ for all $h\in H$. Then, $1\in\{B\}$ implies $\{S\}$ is an $\mathrm{ADS}(2u,u+1,(u+1)/2,(u+3)/2)$, and $1\notin\{B\}$ implies $\{S\}$ is an $\mathrm{ADS}(2u,u-1,(u-1)/2,(u-3)/2)$ in G. In either case, $\boldsymbol{s}$ is an almost balanced perfect $\{0,1\}$ G-array with $G\cong\mathbb{Z}_2\times\Theta(H)\cong\mathbb{Z}_2\times H$, where $\boldsymbol{s}$ is the G-array that corresponds to the group ring element S.

Proof. Since $|S| = u \pm 1$ then $\boldsymbol{s}$ is almost balanced, and by the definition of a Yamada-Pott $\{0,1\}$ H-array pair

$$\begin{aligned} AA^{(-1)} + BB^{(-1)} &= (u \pm 1)(1) + \lambda(H-1) \\ A &= A^{(-1)} \\ B + B^{(-1)} &= H \pm 1, \end{aligned}$$

where

$$\lambda = \begin{cases} \frac{u+1}{2} & \text{if } |A| = |B| = \frac{u+1}{2}, \\ \frac{u-3}{2} & \text{if } |A| = |B| = \frac{u-1}{2}. \end{cases}$$

Now,

$$\begin{aligned} \sum_{t \in G} C_{\boldsymbol{s}}(t)t = SS^{(-1)} = \ & (A + \omega B)(A + \omega B)^{(-1)} \\ = \ & (A + \omega B)\left(A^{(-1)} + \omega B^{(-1)}\right) \\ = \ & AA^{(-1)} + BB^{(-1)} + \omega\left(AB^{(-1)} + BA^{(-1)}\right) \\ = \ & (u \pm 1)(1) + \lambda(H-1) + \omega\left(AB^{(-1)} + BA^{(-1)}\right) \\ = \ & (u \pm 1)(1) + \lambda(H-1) + \omega\left(AB^{(-1)} + BA\right) \\ = \ & (u \pm 1)(1) + \lambda(H-1) + \omega A\left(B^{(-1)} + B\right) \\ = \ & (u \pm 1)(1) + \lambda(H-1) + \omega A\left(H \pm 1\right) \\ = \ & (u \pm 1)(1) + \lambda(H-1) + \omega\left(|A|H \pm A\right) \\ = \ & (u \pm 1)(1) + \lambda(H-1) + \omega\left(\left(\frac{u \pm 1}{2}\right) H \pm A\right) \\ = \ & \begin{cases} \frac{u+1}{2}(1) + \lambda G + A\omega & \text{if } |A| = |B| = \frac{u+1}{2}, \\ \frac{u+1}{2}(1) + \lambda G + (H - A)\omega & \text{if } |A| = |B| = \frac{u-1}{2}, \end{cases} \end{aligned}$$

where we used the group ring equation $G = H + H\omega$. This shows that for $t \neq 1$ the autocorrelation function of $\boldsymbol{s}$ has the following form

$$C_{\boldsymbol{s}}(t) = \begin{cases} \frac{u+1}{2} \text{ or } \frac{u+1}{2} + 1 & \text{if } |A| = |B| = \frac{u+1}{2}, \\ \frac{u-3}{2} \text{ or } \frac{u-3}{2} + 1 & \text{if } |A| = |B| = \frac{u-1}{2}. \end{cases} \tag{10}$$

Thus, by equations (2), (4), and (10), $\boldsymbol{s}$ is an almost balanced and perfect $\{0,1\}$ G-array. □

The following are a few remarks concerning Theorem 3.

1. The equation $AB^{(-1)} + BA = A(B^{(-1)} + B)$ in the proof of Theorem 3 is allowed only when G is abelian. All other steps in the proof would hold for arbitrary finite groups.

2. The converse to Theorem 3 is not true. That is, having a balanced and perfect $\{0,1\}$ $G = \mathbb{Z}_2 \times H$-array does not guarantee the existence of a Yamada-Pott $\{0,1\}$ H-array pair via reversing the construction in Theorem 3. For example,

$$\boldsymbol{s} = (1,1,0,0,1,1,1,1,0,0,1,0,1,1,0,0,0,0,0,0,1,0,1,0,1,1,1,0,1,0,0,1,0,1,1,1,0,0)^\top$$

is a balanced and perfect $\{0,1\}$ $\mathbb{Z}_{38}$-array obtained by applying the $a_g \to \frac{a_g+1}{2}$ transformation to the $\{-1,1\}$ $\mathbb{Z}_{38}$-array in [3]. Let $S = \sum_{i \in \mathbb{Z}_{38}} s_i i$. Now, $\mathbb{Z}_{38} \cong \mathbb{Z}_2 \times \mathbb{Z}_{19}$ via the map $\phi(i) = (i \pmod 2), (i \pmod{19}))$. Let $\hat{S} = \sum_{i \in \mathbb{Z}_{38}} s_{\phi(i)} \phi(i) = \sum_{i' \in \mathbb{Z}_2 \times \mathbb{Z}_{19}} s_{i'} i'$ be the group ring element corresponding to $(\Phi(s_g))$ in $\mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_{19}]$. Write $\hat{S} = \hat{A} + \hat{B}$, where $\hat{A} = \sum_{i' \in \{0\} \times \mathbb{Z}_{19}} s_{i'} i'$ and $\hat{B} = \sum_{i' \in \{1\} \times \mathbb{Z}_{19}} s_{i'} i'$. Let $\pi : \mathbb{Z}_2 \times \mathbb{Z}_{19} \to \mathbb{Z}_{19}$ be the projection map $\pi((x,y)) = y$. Let $A = \sum_{i' \in \{0\} \times \mathbb{Z}_{19}} s_{\pi(i')} \pi(i')$ and $B = \sum_{i' \in \{1\} \times \mathbb{Z}_{19}} s_{\pi(i')} \pi(i')$. Let $\boldsymbol{a}, \boldsymbol{b}$ be the $\{0,1\}$ $\mathbb{Z}_{19}$-arrays corresponding to A and B. Then,

$$\boldsymbol{a} = (1,1,0,1,1,1,1,1,0,1,1,0,1,0,0,1,0,0,0)^\top,$$
$$\boldsymbol{b} = (0,1,0,0,0,1,1,1,0,0,0,0,1,1,1,0,1,0,0)^\top,$$

and $(\boldsymbol{a}, \boldsymbol{b})$ fails all of the Yamada-Pott $\{0,1\}$ $\mathbb{Z}_{19}$-array pair conditions, i.e. none of $\boldsymbol{a}$ and $\boldsymbol{b}$ is symmetric or skew-symmetric and $|A| \neq |B|$. Observe that the isomorphism $\Theta : \mathbb{Z}_2 \times \mathbb{Z}_{19} \to \langle\omega\rangle H$ maps $(\Phi(s_g))$ to the $\langle\omega\rangle H$-array in Theorem 3, where H is a cyclic group of order 19. Hence, by Lemma 6, reversing the construction in Theorem 3 in this case does not produce a Yamada-Pott $\{0,1\}$ H-array pair. In fact, by an exhaustive computer search, we showed that no Yamada-Pott $\{0,1\}$ $\mathbb{Z}_{19}$-pair exists. Similarly, an exhaustive computer search proved that no Yamada-Pott $\{0,1\}$ $\mathbb{Z}_{17}$-array pair exists. However, a balanced and perfect $\{0,1\}$ $\mathbb{Z}_{34}$-array exists as the Ding-Helleseth-Martinsen class 3 $\mathbb{Z}_2 \times \mathbb{Z}_{17}$-array.

3. There are families of balanced, $\{0,1\}$ $\mathbb{Z}_{2u}$-arrays with perfect autocorrelations that can be used to construct Yamada-Pott $\{0,1\}$ $\mathbb{Z}_u$-array pairs or Szekeres $\{0,1\}$ $\mathbb{Z}_u$-array pairs, see Theorems 5 and 6.

4. When $|A| = |B| = (u+1)/2$, the smaller (larger) correlation value appears at the elements of $H \cup (H - A)\omega$ $(A\omega)$.

5. When $|A| = |B| = (u-1)/2$, the smaller (larger) correlation value appears at the elements of $H \cup A\omega$ $((H - A)\omega)$.

Theorem 4. Replacing A with $H - A$ or B with $H - B$ in Theorem 3 does not alter the Yamada-Pott $\{0,1\}$ H-array pair properties 1 and 2, and yields a perfect and balanced $\{0,1\}$ $\langle\omega\rangle H$-array.

Proof. Let $G = \langle\omega\rangle H$ and (A,B) be a Yamada-Pott $\{0,1\}$ H-array pair. Let $S' = (H - A + \omega B)$. Let $\boldsymbol{s}'$ be the $\{0,1\}$ G-array that corresponds to S'. First, $\boldsymbol{s}'$ is balanced as

$$|S'| = |H - A| + |B| = u - \left(\frac{u \pm 1}{2}\right) + \frac{u \pm 1}{2} = u.$$

Secondly, $H - A$ is symmetric as

$$(H - A)^{(-1)} = (H - A^{(-1)}) = H - A.$$

Now,

$$\begin{aligned} S'(S')^{(-1)} = & \\ & = (H - A + \omega B)(H - A + \omega B)^{(-1)} \\ & = (H - A)(H - A)^{(-1)} + BB^{(-1)} + \omega \left(B(H - A)^{(-1)} + (H - A)B^{(-1)}\right). \end{aligned}$$

Then,

$$\begin{aligned} (H - A)(H - A^{(-1)}) + BB^{(-1)} &= HH - HA^{(-1)} - AH + AA^{(-1)} + BB^{(-1)} \\ &= |H|H - HA - AH + AA^{(-1)} + BB^{(-1)} \\ &= uH - 2|A|H + AA^{(-1)} + BB^{(-1)} \\ &= (u - (u \pm 1))H + AA^{(-1)} + BB^{(-1)} \\ &= \mp H + AA^{(-1)} + BB^{(-1)} \\ &= \mp H + (u \pm 1)(1) + \lambda(H - 1) \\ &= \begin{cases} (\frac{u+1}{2} - 1)H + (u - \frac{u+1}{2} + 1)(1) & \text{if } |A| = |B| = \frac{u+1}{2}, \\ (\frac{u-3}{2} + 1)H + (u - \frac{u-3}{2} - 1)(1) & \text{if } |A| = |B| = \frac{u-1}{2} \end{cases} \\ &= \frac{u-1}{2}H + \frac{u+1}{2}(1), \end{aligned}$$

and

$$\begin{aligned} \omega \left(B(H - A)^{(-1)} + (H - A)B^{(-1)}\right) &= \omega(B + B^{-1})(H - A) = \omega(H \pm 1)(H - A) \\ &= \omega\left(\left(\frac{u \pm 1}{2}\right) H \mp A\right). \end{aligned}$$

By examining $S'S'^{(-1)} = (H - A + \omega B)(H - A + \omega B)^{(-1)}$ we see that for $t \neq 1$ the autocorrelation function of $\boldsymbol{s}'$ has the following form

$$C_{\boldsymbol{s}'}(t) = \frac{u \pm 1}{2}. \tag{11}$$

Thus, by equations (2), (3), and (11) the G-array $\boldsymbol{s}'$ is perfect. The case for

$$S' = A + (H - B)\omega$$

is proven similarly. In this case, the skew-symmetry of $H - B$ follows from

$$\begin{aligned}(H-B)+(H-B)^{(-1)} &= H-B+H-B^{(-1)}\\ &= 2H-(B+B^{(-1)})\\ &= 2H-(H\pm 1)\\ &= H\mp 1.\end{aligned}$$

□

2.2 The Ding-Helleseth-Martinsen $\{0,1\}$ $\mathbb{Z}_2 \times \mathbb{Z}_p^m$-array based Yamada-Pott $\{0,1\}$ $\mathbb{Z}_p^m$-array pairs

A Yamada-Pott $\{0,1\}$ $\mathbb{Z}_p^m$-array pair can be obtained from the array pair located by Ding-Helleseth-Martinsen $\{0,1\}$ $\mathbb{Z}_2 \times \mathbb{Z}_p^m$-array in [7] for two cases, where $p^m \equiv 5 \pmod 8$, $p^m = s^2 + 4t^2$, $s \equiv 1 \pmod 4$ and p is a prime. The two cases are $s = 1$ and $t^2 = 1$. When $t^2 = 1$ we get a Yamada-Pott $\{0,1\}$ $\mathbb{Z}_p^m$-array pair, while in the $s = 1$ case or for any $p^m \equiv 5 \pmod 8$, we get a Szekeres $\{0,1\}$ $\mathbb{Z}_p^m$-array pair. First, we present the case of the Ding-Helleseth-Martinsen family of $s = 1$ locating a Szekeres $\{0,1\}$ $\mathbb{Z}_p^m$-array pair for all $p^m \equiv 5 \pmod 8$.

Theorem 5. For each prime power $q = p^m \equiv 5 \pmod 8$ such that $q = s^2+4t^2 = 1+4t^2$, the Ding-Helleseth-Martinsen $\{0,1\}$ $\mathbb{Z}_2 \times \mathbb{Z}_p^m$-array locates the Szekeres $\{0,1\}$ $\mathbb{Z}_p^m$-array pair $(\boldsymbol{a}, \boldsymbol{b})$, where the sets of 1 indices of $(\boldsymbol{a}, \boldsymbol{b})$ are

$$(A, B) = (C_0^{(4,q,\alpha)} \cup C_1^{(4,q,\alpha)}, C_0^{(4,q,\alpha)} \cup C_3^{(4,q,\alpha)}).$$

Proof. The fact that the Ding-Helleseth-Martinsen $\{0,1\}$ $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)$-array locates the Szekeres $\{0,1\}$ $\Theta(\mathbb{Z}_p^m)$-array pair $(\Theta((a_g)), \Theta((b_g)))$, whose sets of 1 indices are

$$(\Theta(C_0^{(4,q,\alpha)} \cup C_1^{(4,q,\alpha)}), \Theta(C_0^{(4,q,\alpha)} \cup C_3^{(4,q,\alpha)})),$$

follows from the definition of the Ding-Helleseth-Martinsen $\{0,1\}$ $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)$-array for $s = 1$. The result now follows from Lemma 4. □

Next, we show that exactly one of the equivalence classes 1 and 2 Ding-Helleseth-Martinsen family with $t^2 = 1$ locates a $\{0,1\}$ Yamada-Pott $\mathbb{Z}_p^m$-array pair.

Let $q = p^m$ for some prime p and $n, D \in \mathbb{Z}$. Then a representation $nq = x^2 + Dy^2$ for some $x, y \in \mathbb{Z}$ is called a *proper* if $\gcd(q, x) = 1$ [17, p. 35]. When $p \equiv 1 \pmod 4$ there are many representations of q in the form $q = s^2 + 4t^2$ for some $s, t \in \mathbb{Z}$. However there is precisely one proper representation [17, p. 47].

Let $q = p^m = 4\ell + 1$ for some prime p and odd positive integer ℓ, or equivalently, let p be a a prime with $p \equiv 5 \pmod 8$ and $m \in 2\mathbb{Z}_{\geq 0} + 1$. Then the unique proper representation of q has the form $q = s^2 + 4t^2$ with $s \equiv 1 \pmod 4$ and $t \in \mathbb{Z}$, where the sign of t is undetermined [17, p. 51]. Let α be a generator of $\mathbb{F}_q^*$. Then, by Lemma 19 in [17, p. 48]

$$t(\alpha) = \frac{16 \times (0,3)^4_{q,\alpha} - q - 1 - 2s}{8}, \tag{12}$$

where $t^2 = (t(\alpha))^2$.

The integers $(i,j)^d_{q,\alpha} = |(C_i^{(d,q,\alpha)} + 1) \cap C_j^{(d,q,\alpha)}|$ are called the *cyclotomic numbers of order d* with respect to $\mathbb{F}_q$ and α such that $\mathbb{F}_q^* = \langle \alpha \rangle$. The following lemma is needed to establish our results.

Lemma 7. Let p be a prime, $p \equiv 5 \pmod 8$, $q = p^m$, and $m \in 2\mathbb{Z}_{\geq 0} + 1$. Let $q = s^2 + 4t^2$ be the unique proper representation of q. Let

$$\begin{aligned} (A_1, B_1) &= (C_0^{(4,q,\alpha)} \cup C_1^{(4,q,\alpha)}, C_1^{(4,q,\alpha)} \cup C_3^{(4,q,\alpha)}), \\ (A_2, B_2) &= (C_0^{(4,q,\alpha)} \cup C_2^{(4,q,\alpha)}, C_2^{(4,q,\alpha)} \cup C_3^{(4,q,\alpha)}), \end{aligned}$$

where $\mathbb{F}_q^* = \langle \alpha \rangle$ and $t(\alpha)$ is as in equation (12). Then

$$|A_1 \cap (A_1 + x)| + |B_1 \cap (B_1 + x)| = \begin{cases} A + 4E + 2B + D = \frac{q - t(\alpha) - 2}{2} & \text{if } x^{-1} \in C_0^{(4,q,\alpha)} \cup C_2^{(4,q,\alpha)}, \\ 4A + 2E + C + D = \frac{q + t(\alpha) - 4}{2} & \text{if } x^{-1} \in C_1^{(4,q,\alpha)} \cup C_3^{(4,q,\alpha)}, \end{cases} \tag{13}$$

and

$$|A_2 \cap (A_2 + x)| + |B_2 \cap (B_2 + x)| = \begin{cases} 4A + 2E + B + C = \frac{q - t(\alpha) - 4}{2} & \text{if } x^{-1} \in C_0^{(4,q,\alpha)} \cup C_2^{(4,q,\alpha)}, \\ 4E + 2D + A + B = \frac{q + t(\alpha) - 2}{2} & \text{if } x^{-1} \in C_1^{(4,q,\alpha)} \cup C_3^{(4,q,\alpha)}, \end{cases} \tag{14}$$

where

$$\begin{aligned} A &= \frac{q - 7 + 2s}{16}, \\ B &= \frac{q + 1 + 2s - 8t(\alpha)}{16}, \\ C &= \frac{q + 1 - 6s}{16}, \\ D &= \frac{q + 1 + 2s + 8t(\alpha)}{16}, \\ E &= \frac{q - 3 - 2s}{16}. \end{aligned}$$

Proof. This result is proven in the proof of Theorem 3.1 in [6]. (There are two typos in equation (5) in [6]; "$\frac{q-2-t}{2}$" and "$\frac{q-4+t}{2}$" should be "$\frac{q-4-t}{2}$" and "$\frac{q-2+t}{2}$" respectively. Equation (5) in [6] is equation (14) here.) ☐

Theorem 6. For $i = 1, 2$, let $(\boldsymbol{a}_i, \boldsymbol{b}_i)$ be $\{0,1\}$ $\mathbb{Z}_p^m$-pair whose sets of 1 indices are (A_i, B_i) in Lemma 7. Then $(\boldsymbol{a}_i, \boldsymbol{b}_i)$ is a Yamada-Pott $\{0,1\}$ $\mathbb{Z}_p^m$-pair if and only if $t(\alpha) = (-1)^{i+1}$, where $t(\alpha)$ is as in equation (12). Hence, exactly one of the $(\boldsymbol{a}_i, \boldsymbol{b}_i)$ is a Yamada-Pott $\mathbb{Z}_p^m$-array pair.

Proof. First, (A_1, B_1) is a 2-SDS$(q; (q-1)/2, (q-1)/2, (q-3)/2)$ if and only if

$$\frac{q - t(\alpha) - 2}{2} = \frac{q + t(\alpha) - 4}{2} = \frac{q-3}{2} \iff t(\alpha) = 1,$$

and (A_2, B_2) is a 2-SDS$(q; (q-1)/2, (q-1)/2, (q-3)/2)$ if and only if

$$\frac{q - t(\alpha) - 4}{2} = \frac{q + t(\alpha) - 2}{2} = \frac{q-3}{2} \iff t(\alpha) = -1.$$

Hence, the choice of field generator α determines which pair is the supplementary difference set as $(0,3)^4_{q,\alpha}$ is a function of α. To prove the symmetry of B_1 and the skew-symmetry of A_1 first observe that $q = s^2 + 4$, with $s \equiv 1 \pmod 4$ implies $q = 8j + 5$ for some $j \in \mathbb{Z}^{\geq 0}$. Since

$$-1 = \alpha^{\frac{q-1}{2}} = \alpha^{4j+2},$$

we have

$$-C_1^{(4,q,\alpha)} = \alpha^{4j+2}\alpha C_0^{(4,q,\alpha)} = \alpha^3 C_0^{(4,q,\alpha)} = C_3^{(4,q,\alpha)},$$

and

$$-C_0^{(4,q,\alpha)} = \alpha^{4j+2} C_0^{(4,q,\alpha)} = C_2^{(4,q,\alpha)}.$$

Then

$$B_1^{(-1)} = (C_1^{(4,q,\alpha)} + C_3^{(4,q,\alpha)})^{(-1)} = -C_1^{(4,q,\alpha)} - C_3^{(4,q,\alpha)} = C_3^{(4,q,\alpha)} + C_1^{(4,q,\alpha)} = B_1,$$

and B_1 is symmetric. Moreover,

$$A_1^{(-1)} = (C_0^{(4,q,\alpha)} + C_1^{(4,q,\alpha)})^{(-1)} = -C_0^{(4,q,\alpha)} - C_1^{(4,q,\alpha)} = C_2^{(4,q,\alpha)} + C_3^{(4,q,\alpha)}. \quad (15)$$

Now, equation (15) implies $\{A_1\} \cap \{A_1^{(-1)}\} = \emptyset$, and $A_1 + A_1^{(-1)} = \mathbb{Z}_p^m - 0$. Hence, A_1 is skew-symmetric. The symmetry of A_2 and the skew-symmetry of B_2 are proven similarly. The result now follows from Theorem 1. □

Let $C_{i,j,l}$ in Section 1.4 be the sets of 1 indices of pairs of $\mathbb{Z}_p^m$-arrays. It is easy to check that the equivalence classes of pairs of $\mathbb{Z}_p^m$-arrays whose sets of 1 indices are $C_{0,1,3}$, $C_{0,2,3}$, and $C_{1,0,3}$ constitute all equivalence classes of all possible pairs of $\mathbb{Z}_p^m$-arrays whose sets of 1 indices have the form $C_{i,j,l}$. Hence, Theorems 5 and 6 cover all equivalence classes of all possible such $\mathbb{Z}_p^m$-arrays.

The following corollary provides two equivalent conditions to the equivalence class i Ding-Helleseth-Martinsen family of $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)$-array $\boldsymbol{s}_i$ being perfect.

Corollary 1. Let $(\boldsymbol{a}_i, \boldsymbol{b}_i)$ and $t(\alpha)$ be as in Theorem 6. Let $\Theta(\mathbb{Z}_2) = \langle \omega \rangle$, and $S_i = \Theta(A_i)+\Theta(B_i)\omega$ be the set of 1 indices of the equivalence class i Ding-Helleseth-Martinsen family of $\{0,1\}$ $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)$-array $\boldsymbol{s}_i$. Then the following are equivalent:

(i) The $\{0,1\}$ $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)$-array $\boldsymbol{s}_i$ is perfect.

(ii) $(\boldsymbol{a}_i, \boldsymbol{b}_i)$ is a Yamada-Pott $\{0,1\}$ $\mathbb{Z}_p^m$-array pair.

(iii) $t(\alpha) = (-1)^{i+1}$.

Proof. The equivalence of (ii) and (iii) follows from Theorem 6. $(ii) \implies (i)$ follows from Theorem 3. To prove $(i) \implies (ii)$, we already proved in the proof of Theorem 6 that $\boldsymbol{a}_1$ and $\boldsymbol{b}_2$ are skew-symmetric and $\boldsymbol{b}_1$ and $\boldsymbol{a}_2$ are symmetric. So, it suffices to show that $(\Theta(\boldsymbol{a}_i), \Theta(\boldsymbol{b}_i))$ is a Legendre pair. By the definition in equation (2), $\boldsymbol{s}_i$ is perfect implies

$$C_{\boldsymbol{s}_i}(t) = \frac{q-1}{2} \quad \text{or} \quad \frac{q-3}{2} \quad \text{if} \quad t \neq 1. \tag{16}$$

Now,

$$\begin{aligned}\sum_{t \in \Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)} C_{\boldsymbol{s}_i}(t)t = S_i S_i^{(-1)} &= (\Theta(A_i) + \omega\Theta(B_i))\,(\Theta(A_i) + \omega\Theta(B_i))^{(-1)} \\ &= (\Theta(A_i) + \omega\Theta(B_i))\left(\Theta(A_i)^{(-1)} + \omega\Theta(B_i)^{(-1)}\right).\end{aligned}$$

Then,

$$S_i S_i^{-1} = \Theta(A_i)\Theta(A_i)^{(-1)} + \Theta(B_i)\Theta(B_i)^{(-1)} + \omega\left(\Theta(A_i)\Theta(B_i)^{(-1)} + \Theta(A_i)^{(-1)}\Theta(B_i)\right). \tag{17}$$

The isomorphism $\Theta : \mathbb{Z}_2 \times \mathbb{Z}_p^m \to \Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)$ extends linearly to an isomorphism of $\mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_p^m]$ and $\mathbb{Z}[\Theta(\mathbb{Z}_2 \times \mathbb{Z}_p^m)]$. If $(\boldsymbol{a}_i, \boldsymbol{b}_i)$ is not a Legendre pair then by equations (13) and (14)

$$A_i(A_i)^{(-1)} + B_i(B_i)^{(-1)}$$

has terms whose coefficients are equal to $(q-5)/2$. Then equation (17) implies

$$\Theta(A_i)\Theta(A_i)^{(-1)} + \Theta(B_i)\Theta(B_i)^{(-1)}$$

has terms whose coefficients are $(q-5)/2$, and this contradicts equation (16). ☐

By establishing $(i) \iff (iii)$ in Corollary 1 we also solved the second of the proposed two open problems at the end of Section 3 in [6]. As far as we know this problem has been open until now.

The second part of Theorem 3.1 of [1] states that the $\mathbb{Z}_p^m$-array pair $(\boldsymbol{a}, \boldsymbol{b})$ with sets of 1 indices $(C_0^{(4,q,\alpha)} \cup C_1^{(4,q,\alpha)}, C_0^{(4,q,\alpha)} \cup C_2^{(4,q,\alpha)})$ satisfies the Legendre $\mathbb{Z}_p^m$-array pair condition. This is not always true. On page 130 of [13], Pott incorrectly credits [1] for this theorem (as it works if and only if $t(\alpha) = -1$). Nevertheless, this does not impact the main theme of [1] on dicyclic designs. The following corollary corrects the second part of Theorem 3.1 of [1].

Corollary 2. Let $t(\alpha)$ and $q = p^m = s^2 + 4t(\alpha)^2$ be as in equation (12). Then, the $\mathbb{Z}_p^m$-array pair $(\boldsymbol{a}, \boldsymbol{b})$ with sets of 1 indices $D_1 = C_0^{(4,q,\alpha)} \cup C_1^{(4,q,\alpha)}$, and $D_2 = C_0^{(4,q,\alpha)} \cup C_2^{(4,q,\alpha)}$ satisfies the Legendre $\mathbb{Z}_p^m$-array pair condition if and only if $t(\alpha) = -1$. Moreover, $\boldsymbol{a}$ is skew-symmetric and $\boldsymbol{b}$ is symmetric.

Proof. Let A_2, B_2 and $\boldsymbol{a}_2$, $\boldsymbol{b}_2$ be as in Theorem 6. Then $\boldsymbol{a}_2$ is symmetric and $\boldsymbol{b}_2$ is skew-symmetric. Observe that $(D_1, D_2) = (\alpha^2 B_2, \alpha^2 A_2)$. Hence, $\mathbb{Z}_p^m$-array pair $(\boldsymbol{b}_2, \boldsymbol{a}_2)$ is equivalent to $(\boldsymbol{a}, \boldsymbol{b})$. Thus, $(\boldsymbol{a}, \boldsymbol{b})$ is a Legendre $\mathbb{Z}_p^m$-array pair if and only if $t(\alpha) = -1$. By Lemma 5, $(D_1, D_2) = (\alpha^2 B_2, \alpha^2 A_2)$ implies $\boldsymbol{a}$ is skew-symmetric and $\boldsymbol{b}$ is symmetric. □

2.3 The Sidelnikov-Lempel-Cohn-Eastman $\mathbb{Z}_{q-1}$-array based Yamada-Pott $\{0,1\}$ $\mathbb{Z}_{(q-1)/2}$-array pairs

An interesting fact about the Sidelnikov-Lempel-Cohn-Eastman $\{0,1\}$ $\mathbb{Z}_{q-1}$-array and the Yamada Yamada-Pott $\{0,1\}$ $\mathbb{Z}_{(q-1)/2}$-array pair is that each pair can be obtained from the other.

Theorem 7. For $q \geq 7$ and $q \equiv 3 \pmod 4$ let (A_1, B_1) and $(A_2 \cup B_2)$ be the pair of sets of 1 indices of the Yamada Yamada-Pott $\{0,1\}$ $\mathbb{Z}_{(q-1)/2}$-array pair and the set of 1 indices of the Sidelnikov-Lempel-Cohn-Eastman $\{0,1\}$ $\mathbb{Z}_{q-1}$-array, where

$$\begin{aligned}
A_1 &= \left\{\log_\alpha x \ (\mathrm{mod}\ \tfrac{q-1}{2}) \mid x \in (C_0^2 + 1) \cap C_0^2\right\}, \\
B_1 &= \left\{\log_\alpha x \ (\mathrm{mod}\ \tfrac{q-1}{2}) \mid x \in (C_0^2 - 1) \cap C_0^2\right\}, \\
A_2 &= \left\{\log_\alpha x \ (\mathrm{mod}\ \tfrac{q-1}{2}) \mid x \in (C_1^2 - 1) \cap C_0^2\right\}, \\
B_2 &= \left\{\log_\alpha x \ (\mathrm{mod}\ \tfrac{q-1}{2}) \mid x \in (C_1^2 - 1) \cap C_1^2\right\}.
\end{aligned}$$

Then, $A_1 = B_2$, and $B_1 = \mathbb{Z}_{\frac{q-1}{2}} \setminus A_2$.

Proof. Observe that

$$\alpha^{\frac{q-1}{2}}[(C_0^2 + 1) \cap C_0^2] = (\alpha^{\frac{q-1}{2}} C_0^2 + \alpha^{\frac{q-1}{2}}) \cap \alpha^{\frac{q-1}{2}} C_0^2.$$

Then, $q \equiv 3 \pmod 4$ implies $\alpha^{(q-1)/2} = -1 \notin C_0^2$ giving

$$\alpha^{\frac{q-1}{2}}[(C_0^2 + 1) \cap C_0^2] = (C_1^2 - 1) \cap C_1^2.$$

After taking the discrete logarithm and reducing modulo $(q-1)/2$, we get $A_1 = B_2$. Since $\mathbb{F}_q = \{0\} \cup C_0^2 \cup C_1^2$ is a partitioning of $\mathbb{F}_q$ and $\phi(x) = x-1$ is a one-to-one function from $\mathbb{F}_q$ to $\mathbb{F}_q$ we get

$$\mathbb{F}_q = \{-1\} \cup (C_0^2 - 1) \cup (C_1^2 - 1) \tag{18}$$

as another partitioning of $\mathbb{F}_q$. Now, by equation (18) and the fact that $-1 \notin C_0^2$, we get

$$C_0^2 = C_0^2 \cap [(C_0^2 - 1) \cup (C_1^2 - 1)] = (C_0^2 \cap (C_0^2 - 1)) \cup (C_0^2 \cap (C_1^2 - 1))$$

as a partitioning of C_0^2. Then, we get the set equations

$$2\mathbb{Z}_{\frac{q-1}{2}} = \log_\alpha(C_0^2) = \log_\alpha[(C_0^2 \cap (C_0^2 - 1)) \cup (C_0^2 \cap (C_1^2 - 1))] =$$
$$\log_\alpha[(C_0^2 \cap (C_0^2 - 1))] \cup \log_\alpha[(C_0^2 \cap (C_1^2 - 1))] = 2[B_1 \cup A_2]$$

as

$$(C_0^2 \cap (C_0^2 - 1)) \cap (C_0^2 \cap (C_1^2 - 1)) = \emptyset.$$

Hence,

$$\mathbb{Z}_{\frac{q-1}{2}} = \frac{1}{2}\log_\alpha(C_0^2) = B_1 \cup A_2.$$

It is also clear that $A_2 \cap B_1 = \emptyset$. Thus,

$$B_1 = \mathbb{Z}_{\frac{q-1}{2}} \setminus A_2 \quad \text{and} \quad A_2 = \mathbb{Z}_{\frac{q-1}{2}} \setminus B_1.$$

□

Next, we locate an almost balanced perfect $\{0,1\}$ $\mathbb{Z}_{q-1}$-array pair based on a family of $\{0,1\}$ $\mathbb{Z}_{(q-1)/2}$-array pairs. In fact, this result is presented partially in [6], as Theorem 4.1. By Lemma 6 it suffices to locate an almost balanced perfect $\{0,1\}$ $\Theta(\mathbb{Z}_{q-1})$-array.

Theorem 8. Let $q \geq 7$ and $q = p^m \equiv 3 \pmod 4$, α be a generator of $\mathbb{F}_q^*$. Let

$$A = \left\{\log_\alpha x \pmod{\tfrac{q-1}{2}} \mid x \in (C_0^2 - 1) \cap C_0^2\right\},$$
$$B = \left\{\log_\alpha x \pmod{\tfrac{q-1}{2}} \mid x \in (C_0^2 - 1) \cap C_1^2\right\}$$

be the pair of sets of 1 indices of the $\{0,1\}$ $\mathbb{Z}_{(q-1)/2}$-array $(\boldsymbol{a}, \boldsymbol{b})$ pair. Then $(\boldsymbol{a}, \boldsymbol{b})$ is a Yamada-Pott $\{0,1\}$ $\mathbb{Z}_{(q-1)/2}$-array pair. Let $\langle\omega\rangle = \Theta(\mathbb{Z}_2)$. Then $\Theta(A) + \Theta(B)\omega \in \mathbb{Z}[\Theta(\mathbb{Z}_2 \times \mathbb{Z}_{(q-1)/2})]$ corresponds to an almost balanced perfect $\{0,1\}$ $\Theta(\mathbb{Z}_2 \times \mathbb{Z}_{(q-1)/2})$-array.

Proof. Let $A' = \sum_{g \in A/2} g$ and $B' = \sum_{g \in (B-1)/2} g$, where $A', B' \in \mathbb{Z}[\mathbb{Z}_{(q-1)/2}]$ and $(\boldsymbol{a}', \boldsymbol{b}')$ be the corresponding $\mathbb{Z}_{(q-1)/2}$-array pair. In Theorem 4.1 of [6] it is shown that

$$A'(A')^{(-1)} + B'(B')^{(-1)} = \frac{q-7}{4}(\mathbb{Z}_{\frac{q-1}{2}} - 0) + \frac{q-3}{2}(0).$$

Now, since $(\boldsymbol{a}', \boldsymbol{b}')$ and $(\boldsymbol{a}, \boldsymbol{b})$ are equivalent $\mathbb{Z}_{(q-1)/2}$-array pairs, we get that $(\boldsymbol{a}, \boldsymbol{b})$ is a Legendre $\{0,1\}$ $\mathbb{Z}_{(q-1)/2}$-array pair. Next, we show that $\boldsymbol{a}$ is a symmetric $\mathbb{Z}_{(q-1)/2}$-array. For a set $A \subseteq \mathbb{Z}_{(q-1)/2}$ let $-A = \{x \mid -x \in A\}$. For any $x \in (C_0^2 - 1) \cap C_0^2$ we have

$$x = \alpha^{2i} - 1 = \alpha^{2j}, \tag{19}$$

for some $i, j \in \mathbb{Z}$. Multiplying both sides of equation (19) by $x^{-1} = \alpha^{-2j}$ yields

$$\alpha^{2i-2j} - \alpha^{-2j} = 1$$

or

$$\alpha^{-2j} = \alpha^{2i-2j} - 1.$$

Then, $x^{-1} \in (C_0^2 - 1) \cap C_0^2 = A$ implying $-2j \in A$. Hence, if $2j \in A$ then $2j \in -A$. Since $|(-A)| = |A|$ we get $A = -A$. Finally, we show that $\boldsymbol{b}$, equivalently B is skew-symmetric. By the definition of B, any $x \in (C_0^2 - 1) \cap C_1^2$ satisfies

$$x = \alpha^{2i} - 1 = \alpha^{2j+1}, \tag{20}$$

for some $i, j \in \mathbb{Z}$. By multiplying both sides of equation (20) with $x^{-1} = \alpha^{-(2j+1)}$ we see that

$$\alpha^{2i-2j-1} - \alpha^{-(2j+1)} = 1.$$

By rearranging terms we get

$$\alpha^{-(2j+1)} = \alpha^{2i-2j-1} - 1,$$

and so $x^{-1} \in (C_1^2 - 1) \cap C_1^2$. Hence, $x^{-1} \notin C_0^2 - 1$, and $-2j - 1 \notin B$. Thus, if $b = 2j + 1 \in B$, then $-b \notin B$ giving that $B \cap (-B) = \emptyset$. Now, $q \equiv 3 \pmod 4$ implies $\alpha^{(q-1)/2} = -1 \notin C_0^2$. Then,

$$\alpha^{\frac{q-1}{2}}[(C_0^2 - 1) \cap C_1^2] = (C_1^2 + 1) \cap C_0^2$$

and consequently $|(C_0^2 - 1) \cap C_1^2| = |(C_1^2 + 1) \cap C_0^2|$. Now, by equation (18) and the fact that $-1 \in C_1^2$, we get

$$C_1^2 = C_1^2 \cap [(C_0^2 - 1) \cup (C_1^2 - 1)] = \{-1\} \cup (C_1^2 \cap (C_0^2 - 1)) \cup (C_1^2 \cap (C_1^2 - 1))$$

as a partitioning of C_1^2. Then, the set equations

$$\begin{aligned} &2\mathbb{Z}_{\frac{q-1}{2}} + 1 = \log_\alpha(C_1^2) = \log_\alpha[-1] \cup \log_\alpha[(C_1^2 \cap (C_0^2 - 1)) \cup (C_1^2 \cap (C_1^2 - 1))] = \\ &\log_\alpha[-1] \cup \log_\alpha[(C_1^2 \cap (C_0^2 - 1))] \cup \log_\alpha[(C_1^2 \cap (C_1^2 - 1))] \end{aligned}$$

gives a partitioning of $2\mathbb{Z}_{(q-1)/2} + 1$ as

$$(C_1^2 \cap (C_0^2 - 1)) \cap (C_1^2 \cap (C_1^2 - 1)) = \emptyset,$$

$-1 \notin (C_1^2 \cap (C_0^2 - 1))$ and $-1 \notin (C_1^2 \cap (C_1^2 - 1))$. Since $\gcd((q-1)/2, 2) = 1$, $\phi(x) = 2x+1$ is an automorphism of $\mathbb{Z}_{(q-1)/2}$. Then

$$\left(2\mathbb{Z}_{\frac{q-1}{2}} + 1\right) (\bmod \ \frac{q-1}{2}) = \left(\log_\alpha[-1] \cup \log_\alpha[(C_1^2 \cap (C_0^2 - 1))] \cup \log_\alpha[(C_1^2 \cap (C_1^2 - 1))]\right) (\bmod \ \frac{q-1}{2})$$

is a partitioning of $\mathbb{Z}_{(q-1)/2}$. This implies that $|B| = |(C_0^2 - 1) \cap C_1^2|$. By part b of Lemma 6 in [17, p. 30], $|(C_0^2 - 1) \cap C_1^2| = |(C_1^2 + 1) \cap C_0^2| = (q-3)/4$. Hence, $|B| = (q-3)/4$. We also have $|B| = |(-B)|$ and $B \cap (-B) = \emptyset$, so $B \cup (-B) = \mathbb{Z}_{(q-1)/2} \backslash 0$. Now, the result follows from Lemma 2. □

While it is believed that a Legendre $\{0,1\}$ $\mathbb{Z}_n$-array pair exists for all odd n, the existence of Yamada-Pott $\{0,1\}$ $\mathbb{Z}_n$-array pairs or $\{0,1\}$ $\mathbb{Z}_n$-array pairs $(\boldsymbol{a}, \boldsymbol{b})$ such that both $\boldsymbol{a}$ and $\boldsymbol{b}$ are symmetric or skew-symmetric has not received as much attention. Table 1 shows the existence and non-existence of $\{0,1\}$ Yamada-Pott $\mathbb{Z}_n$-array pairs. The comment column describes either how the pair is generated or how we have shown nonexistence. "Computer search" means the existence or non-existence of a Yamada-Pott $\{0,1\}$ $\mathbb{Z}_n$-array was proven by an exhaustive computer search. Under the "Exist?" column a "Y" or "N" means yes or no. Our computer search was based on going through all possible pairs of $\{0,1\}$ sequences, $\boldsymbol{a}, \boldsymbol{b}$ such that

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = \frac{n+1}{2}$$

and screening out the pairs that formed a Legendre pair. At the end of the search, for each found $\{0,1\}$ Legendre $\mathbb{Z}_n$-array pair $(\boldsymbol{a}, \boldsymbol{b})$, we checked for the symmetry and skew symmetry of $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

Table 2 shows the existence of a Legendre $\{0,1\}$ $\mathbb{Z}_n$-array pair for all possible combinations of $\boldsymbol{a}$ and $\boldsymbol{b}$ being symmetric, skew-symmetric and neither symmetric nor skew-symmetric. The number at the top of each column is n. The first two columns describe the attributes of $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. In the first two columns "N" means neither symmetric nor skew-symmetric, "Sk" means skew-symmetric and "S" means symmetric. For each cell that is in a column with an integer at the top, "E" and "NE" mean exists and does not exist respectively.

Exhaustive searches proved that no balanced, perfect $\{0,1\}$ $\mathbb{Z}_{54}$-array exists, on two different supercomputers, with different programs [11]. This is consistent with our computer searches as finding a Yamada-Pott $\{0,1\}$ $\mathbb{Z}_{27}$-array pair would imply a perfect balanced $\{0,1\}$ $\mathbb{Z}_{54}$-array by Theorem 3.

We end this section with a couple of comments.

1. In [13], on page 130, it is claimed that a Yamada-Pott $\{0,1\}$ $\mathbb{Z}_{37}$-array pair exists. This is false as it originated from the mistake in part 2 of Theorem 3.1 in [1]. The

Table 1: The existence of Yamada-Pott $\{0,1\}$ $\mathbb{Z}_n$-array pairs

n	Exist?	Comment
3	Y	Theorem 8 with $q = 2(3)+1 = 7$
5	Y	Theorem 8 with $q = 2(5)+1 = 11$
7	N	Computer search
9	Y	Theorem 8 with $q = 2(9)+1 = 19$
11	Y	Theorem 8 with $q = 2(11)+1 = 23$
13	Y	Theorem 8 with $q = 2(13)+1 = 27$
15	Y	Theorem 8 with $q = 2(15)+1 = 31$
17	N	Computer search
19	N	Computer search
21	Y	Theorem 8 with $q = 2(21)+1 = 43$
23	Y	Theorem 8 with $q = 2(23)+1 = 47$
25	N	Computer search
27	N	Computer search
29	Y	Theorem 8 with $q = 2(29)+1 = 59$
31	N	Computer search

case $n = 37$ with $q = 2*37+1 = 75$ is not a prime power. However, a symmetric Paley $\{0,1\}$ $\mathbb{Z}_{37}$-array pair and a skew-symmetric Szekeres $\{0,1\}$ $\mathbb{Z}_{37}$-array pair exist. The first example of a Yamada-Pott $\{0,1\}$ $\mathbb{Z}_q$-array pair that arises from Theorem 6 is at $q = 15^2+4 = 229$, where $n = 2q+1 = 459 = 27*17$ is not a prime power.

2. Exhaustive searches proved that no Legendre $\{0,1\}$ $\mathbb{Z}_7$-array pair $(\boldsymbol{a}, \boldsymbol{b})$, where $\boldsymbol{a}$ is symmetric, and no Legendre $\{0,1\}$ $\mathbb{Z}_{17}$-array pair $(\boldsymbol{a}, \boldsymbol{b})$, where $\boldsymbol{a}$ is skew-symmetric exists. Exhaustive searches found a Legendre $\{0,1\}$ $\mathbb{Z}_n$-array pair $(\boldsymbol{a}, \boldsymbol{b})$, where at least one of $\boldsymbol{a}$ and $\boldsymbol{b}$ is symmetric or skew-symmetric for each $n \leq 21$, see Table 2.

2.4 An inequivalent Legendre $\{-1,1\}$ $\mathbb{Z}_{57}$-array pair

By using a heuristic computer search the only known example of a Legendre $\{-1,1\}$ $\mathbb{Z}_{57}$-array pair was found in [5]. This resulted in the construction of a 116×116 Hadamard matrix via Theorem 2. The Legendre $\{-1,1\}$ $\mathbb{Z}_{57}$-array pair found is given by

$$\boldsymbol{a}_1 = (-++-+-+-++-++----+-+++++-+++--+++-+-----+-+--+-+++--+-+--)^\top,$$

$$\boldsymbol{b}_1 = (--++--++-++++-++-+++++-----+-+++-++--+-+---+++----++----+)^\top,$$

Table 2: The existence of a Legendre $\{0,1\}$ $\mathbb{Z}_n$-array pair for all possible combinations of $\boldsymbol{a}$ and $\boldsymbol{b}$ being symmetric, skew-symmetric and neither symmetric nor skew-symmetric

Type		n								
A	B	5	7	9	11	13	15	17	19	21
N	N	E	E	E	E	E	E	E	E	E
N	S	E	NE	E	E	E	E	E	E	E
N	Sk	E	E	E	E	E	E	NE	E	E
S	S	E	NE	NE	NE	E	NE	E	NE	NE
S	Sk	E	NE	E	E	E	E	NE	NE	E
Sk	Sk	E	E	NE	E	E	NE	NE	E	NE

where $-$, $+$ are used for -1, 1, and commas are deleted to save space.
This pair can be shown to satisfy the condition given by Definition 5. The distributions of the autocorrelations of A and B are

$$(-11)^2(-7)^{12}(-3)^{10}(1)^{20}(5)^{12},$$

and

$$(-7)^{12}(-3)^{20}(1)^{10}(5)^{12}(9)^2.$$

By using cyclotomy we found a Legendre $\{-1,1\}$ $\mathbb{Z}_3 \times \mathbb{Z}_{19}$-array pair $(\boldsymbol{x}, \boldsymbol{y})$ that can be used to construct a Legendre $\{-1,1\}$ $\mathbb{Z}_{57}$-array pair that is not equivalent to the previously known $\{-1,1\}$ Legendre $\mathbb{Z}_{57}$-array pair. This construction is displayed in the next example.

Example 1. Construct $C_i^{(6,19)}$ for $i = 0, 1, \ldots, 5$ for $\alpha = 2$. For this example, we explicitly construct these cosets for $d = 6$, $q = 19$ and $\alpha = 2$. The elements are given by $C_0^{(6,19,2)} = \{1, 7, 11\}$ with the remaining cosets being generated by multiplying $C_0^{(6,19,2)}$ by $\alpha = 2$ and reducing modulo 19. For brevity, we use C_i^6 for $C_i^{(6,19,2)}$. Let

$$X = \left\{\{0\} \times \{0, C_0^6, C_1^6, C_2^6\}\right\} \cup \left\{\{1\} \times \{C_0^6, C_2^6, C_3^6, C_4^6\}\right\} \cup \left\{\{2\} \times \{C_3^6, C_4^6\}\right\},$$

and

$$Y = \left\{\{0\} \times \{0, C_0^6, C_4^6, C_5^6\}\right\} \cup \left\{\{1\} \times \{C_0^6, C_3^6, C_5^6\}\right\} \cup \left\{\{2\} \times \{C_0^6, C_1^6, C_3^6\}\right\}.$$

Then, the Legendre $\{-1,1\}$ $\mathbb{Z}_3 \times \mathbb{Z}_{19}$-array pair $(\boldsymbol{x}, \boldsymbol{y})$ obtained by letting

$$x_i = \begin{cases} 1 & \text{if } i \in X, \\ -1 & \text{otherwise,} \end{cases}$$

and

$$y_i = \begin{cases} 1 & \text{if } i \in Y, \\ -1 & \text{otherwise} \end{cases}$$

satisfies Definition 5. The distribution of autocorrelations for $\boldsymbol{x}$ and $\boldsymbol{y}$ are

$$(-7)^{14}(-3)^{12}(1)^{18}(5)^{12},$$

$$(-7)^{12}(-3)^{18}(1)^{12}(5)^{14}.$$

The correlation energy for the $\{-1,1\}$ $\mathbb{Z}_3 \times \mathbb{Z}_{19}$ array pair $(\boldsymbol{x}, \boldsymbol{y})$ is $1,112$. To construct a Legendre $\{-1,1\}$ $\mathbb{Z}_{57}$ array pair observe that $\mathbb{Z}_{57} \cong \mathbb{Z}_3 \times \mathbb{Z}_{19}$ via the map $\phi(i) = (i \pmod 3), (i \pmod{19}))$. Then, the $\{-1,1\}$ $\mathbb{Z}_{57}$-array pair $(\phi^{-1}((x_g)), \phi^{-1}((y_g)))$ is a Legendre pair that has the same distribution of autocorrelations and the same correlation energy as those of $(\boldsymbol{x}, \boldsymbol{y})$. The map ϕ^{-1} is constructed as follows. Let $\phi(i) = (i \pmod 3), (i \pmod{19})) = (a, b)$. Then there exists $k_1, k_2 \in \mathbb{Z}^{\geq 0}$ such that $a + 3k_1 = b + 19k_2 = i$, $0 \leq k_1 \leq 19$, and $0 \leq k_2 \leq 3$. Then, $3k_1 + (a - b) = 19k_2$ imply $k_2 = (a-b)19^{-1} \pmod 3$ and $k_1 = -(a-b)3^{-1} \pmod{19}$. Now, k_1, k_2 are uniquely determined by the inequalities $0 \leq k_1 \leq 19$, and $0 \leq k_2 \leq 3$. Hence, $\phi^{-1}(a, b) = a + 3k_1 = b + 19k_2 = i$. This gives us

$$\boldsymbol{a}_2 = (++-+++++++------++---+---+--+-++-+-+-+-+--++-++--++----++)^\top,$$

$$\boldsymbol{b}_2 = (+++----++-++-+++----+---+-+---++--+-++-+-+---++-++++--+-+)^\top.$$

The correlation energy for the Legendre $\{-1,1\}$ $\mathbb{Z}_{57}$-array pair in [5] is $1,240$. Thus $(\boldsymbol{a}_2, \boldsymbol{b}_2)$ is not equivalent to $(\boldsymbol{a}_1, \boldsymbol{b}_1)$.

We propose developing theoretical and computational methods for finding Legendre $\{-1,1\}$ $\mathbb{Z}_n$-array pairs by using cyclotomic cosets as in Example 1 as a future research direction.

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